Who is More Bayesian: Humans or ChatGPT?*

Tianshi Mu, Tsinghua University
Pranjal Rawat, Georgetown University
John Rust, Georgetown University †
Chengjun Zhang, Georgetown University
Qixuan Zhong, Georgetown University

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Abstract

We compare the performance of human and artificially intelligent (AI) decision makers in simple binary classification tasks where the optimal decision rule is given by Bayes Rule. We reanalyze choices of human subjects gathered from laboratory experiments conducted by El-Gamal and Grether and Holt and Smith. We confirm that while overall, Bayes Rule represents the single best model for predicting human choices, subjects are heterogeneous and a significant share of them make suboptimal choices that reflect judgement biases described by Kahneman and Tversky that include the "representativeness heuristic" (excessive weight on the evidence from the sample relative to the prior) and "conservatism" (excessive weight on the prior relative to the sample). We compare the performance of AI subjects gathered from recent versions of large language models (LLMs) including several versions of ChatGPT. These general-purpose generative AI chatbots are not specifically trained to do well in narrow decision making tasks, but are trained instead as "language predictors" using a large corpus of textual data from the web. We show that ChatGPT is also subject to biases that result in suboptimal decisions. However we document a rapid evolution in the performance of ChatGPT from sub-human performance for early versions (ChatGPT 3.5) to superhuman and nearly perfect Bayesian classifications in the latest versions (ChatGPT 4o).

Keywords: Bayes rule, decision making, statistical decision theory, win and loss functions, learning, classification, machine learning, artificial intelligence, large language models, Bayesian inference, maximum likelihood, heterogeneity, mixture models, Estimation-Classification (EC) algorithm, binary logit model, structural models

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†**Corresponding Author:** Department of Economics, Georgetown University, jr1393@georgetown.edu.

1 Introduction

We compare the performance of human and artificially intelligent (AI) decision makers in simple binary classification tasks where the optimal decision rule is given by Bayes Rule. AI algorithms such as support vector machines or neural networks can be trained to closely approximate optimal Bayesian decision rules given the relative simplicity of this narrow domain classification task. Machine learning methods have been extended to more difficult real world classification problems where the covariates used to classify outcomes can be very high dimensional (e.g. using mammograms to detect breast cancer). A number of studies have shown these specially trained classifiers can perform at superhuman levels, see e.g. Yoen and Chang (2023).

It is not surprising that humans, whose brains consume only about 20 watts of power, do not outperform special-purpose machine learning algorithms that are trained with large volumes of data to approximate optimal decision rules for specific tasks. Kühl et al. (2022) show that humans possess *general intelligence* whereas most machine learning algorithms are designed to work in narrow domains and will not necessarily make sensible decisions in a huge variety of different (and often unexpected) situations as humans do. As Hutchinson and Meyer (1994) noted, *"From a broader perspective, however, one can argue that optimal solutions are known for a relatively small number of similar, well-specified problems whereas humans evolved to survive in a world filled with a large and diverse set of ill-specified problems. Our 'suboptimality' may be a small price to pay for the flexibility and adaptiveness of our intuitive decision processes."*

Rapid recent improvements in large language models (LLMs) and generative AI suggest that we may be close to the advent of *Artificial general intelligence* (AGI) where general purpose algorithms equal or exceed human performance in solving a wide range of problems even though the algorithms were not specifically trained to do well in specific narrow domain tasks. Generative AI models such as ChatGPT are deep neural networks with billions of parameters that have been trained to predict text, sounds and images using vast databases obtained from the web and other sources. The progress in this area has been breathtaking, and now a variety of LLMs have demonstrated a capability to compete with humans on a wide range of intellectual tasks.[1]

[1]In the paper we will use interchangeably the abbreviations LLM and GPT (for Generative Pretrained Transformer), though the latter is a subset of the former. See section 7 for further discussion and comments on the

Despite the amazingly rapid improvements, the consensus is that LLMs still lack full rationality, including the capability to reason and think creatively the way humans do, and other features associated with intelligence including "consciousness". The review by Maslej et al. (2024) concludes that *"AI has surpassed human performance on several benchmarks, including some in image classification, visual reasoning, and English understanding. Yet it trails behind on more complex tasks like competition-level mathematics, visual commonsense reasoning and planning."*

Our study focuses on statistical decision making, building on a large experimental literature designed to assess the rationality of human subjects in a simple binary classification problem used in dozens of previous experimental studies in economics and psychology. The decision problem is simple enough to have an optimal solution that is easily characterized using statistical decision theory and Bayes Rule. Numerous studies in psychology and behavioral economics conclude that humans make suboptimal decisions due to systematic biases, including "framing" and contextual effects that might be caused by reliance on heuristics to reduce cognitive burden, see e.g. Tversky and Kahneman (1974).

However this conclusion is controversial due to the use of "real world" situations to test decision making (e.g. providing a description of a person and asking them to choose whether they are more likely to be an engineer or a lawyer), since it provides extraneous information that amplifies the potential for framing effects and stereotyping to mislead subjects. Grether (1978) noted that Kahneman and Tversky's experiments had "features that make the applicability of the findings to economic decisions doubtful" and their framing of scenarios created "difficulty of controlling the information given when verbal descriptions or situations are presented. Both of these difficulties could be taken care of by the use of actual balls in urns or book-bag pokerchip setups." (p. 71-72). Cosmides and Tooby (1996) argued that experiments framed in frequentist terms are more likely to generate behavior that conforms to Bayes Rule since "our inductive reasoning mechanisms do embody aspects of a calculus of probability, but they are designed to take frequency information as input and produce frequencies as output." (p. 3).

We find this logic compelling, and build on the experimental design of El-Gamal and Grether (1995) where subjects were shown the outcomes of samples of balls drawn with replacement from one of two bingo cages, A or B, where each contains different

rapidly growing literature that compares human and AI performance over a wide range of domains and problems.

proportions of red and blue balls. The cage used to draw the sample was selected by a random procedure, such as choosing cage A if the throw of 6-sided die is 1 or 2. This induces a credible objective prior probability of selecting cage A. The experimental controls involve different choices of these prior probabilities, the proportions of red and blue balls in the two cages, and number of balls drawn with replacement. Subjects were informed of these parameters and the outcome of the sample drawn from the selected cage. Based on this information subjects chose the cage they believed was more likely to have been selected. The experiments by Holt and Smith (2009) used a similar design but asked subjects to directly report subjective posterior probabilities of cage A.

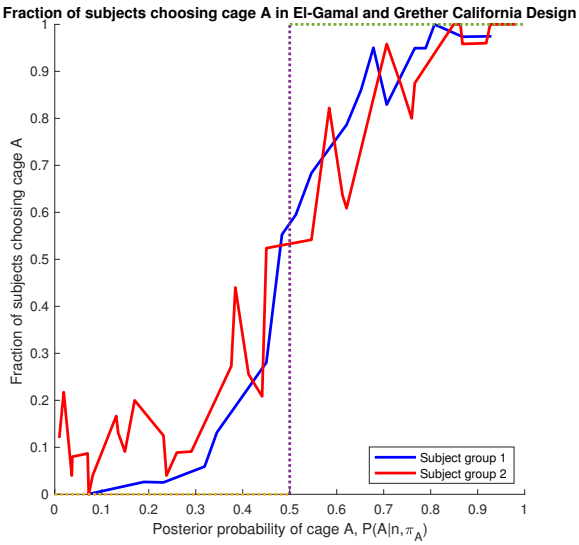
The "textbook" solution to this problem is to use Bayes Rule to compute the posterior probability of A and select it if and only if its posterior probability exceeds 1/2. However the experiments did not presume that subjects actually know Bayes Rule and did not train them or tip them off about how to make "correct" responses. The objective of these studies was to assess the extent to which untrained subjects behave as "intuitive Bayesians" and make instinctive choices that are consistent with Bayes Rule. This should be the case if subjects are *intelligent and greedy* since subjects received bonus payments for choosing the correct bingo cage used to draw the sample. We hypothesize that their behavior should be governed by an *optimal decision rule,* i.e. one that maximizes the probability of selecting the correct bingo cage. It is straightforward to show that choosing according to Bayes Rule is the optimal decision rule for this problem. Thus, a comparison of whether humans or GPT is more "Bayesian" is equivalent to a test of their ability to conceptualize probabilities and optimize.[2]

To preview our key findings, and convince the reader that AI subjects are not automatically superior to human decision makers, consider figure 1. The graph plots the fraction of subjects who chose cage A (y axis) against the true Bayesian posterior probability for cage A (x axis) using the experimental design of El-Gamal and Grether (1995) and El-Gamal and Grether (1999) where subjects were shown the outcome of 6 balls drawn with replacement from one of the randomly selected cages. The thin dotted blue line is the optimal decision rule of a perfect Bayesian decision maker: choose cage B if the posterior probability of cage A is less than 1/2 and choose cage A otherwise. The solid

[2]In the experiments where subjects reported their subjective posterior probability for cage A a more complicated incentive scheme known as a "BDM mechanism" was used that requires subjects to solve an even more difficult continuous optimization problem that we describe in section 4 of the paper.

red and blue lines represent two different groups of subjects: one is a group of human subjects, and another is a group of ChatGPT 4 subjects. As a teaser, we won't reveal which group is which at this point and let the reader try to guess, emphasizing the "Turing test" aspect of our analysis and the possibility that at least the early generations of GPTs may in fact suffer from some of the same foibles as human decision makers. Further it is not immediately obvious which of the subject groups exhibits "better" performance.

Figure 1: Which group of subjects, 1 or 2, are GPT and which are human?



The simple nature of this binary classification problem allows us to obtain an objective measure of "performance" to compare human and GPT subjects. In section 2 we introduce the relevant statistical decision theory and define the *win function,* i.e. the conditional probability of selecting the correct cage implied by any given decision rule. The win function is the complement of the *loss function* in statistical decision theory, so any decision rule that maximizes the win function necessarily minimizes the loss function. Using the win function, we define a simple measure of *efficiency* of a human or GPT decision maker: the ratio of the expected win probability of the decision maker to the optimal win probability implied by Bayes Rule. This is a superior measure of performance than the commonly used notion of *accuracy,* i.e. the fraction of a subject's choices that coincide with Bayesian choices. Our efficiency measure differentiates between "hard cases" (where the Bayesian posterior probability is close to 1/2) and "easy cases" (those where the posterior probability is close to 0 or 1). Among two subjects facing the same

set of trials and whose choices agree with Bayes Rule on the same total number of these trials (so the subjects have equal accuracy), the subject who has a greater fraction of choices that disagree with Bayes Rule on the easy cases will have a lower expected payoff and thus lower efficiency.

Our re-analysis of the human experiments confirms the conclusions of El-Gamal and Grether (1995) and Holt and Smith (2009) that humans, on average, closely resemble Bayesian decision makers. The decision efficiency reached above 96% in both experiments. We detect considerable unobserved heterogeneity in human subjects, with the most efficient type nearly matching Bayesian decision makers.

Next, we document a rapid improvement in the decision efficiency of GPTs over a few generations. Initially, GPT-3.5's decision efficiency is 12.1% lower than that of humans in the binary choice Wisconsin experiment and 21.9% lower in the more challenging Holt-Smith experiment. However, with a few updates, GPT-4's efficiency quickly reaches a level comparable to humans, and GPT-4o surpasses human efficiency by 1% and 3.5% in the respective experiments. We also conducted a limited analysis of the most recently released version of ChatGPT o1, and found that its efficiency is close to 100%, with behavior that closely approximates a perfect Bayesian decision maker.[3]

A key advantage of AI subjects over humans is that AI subjects provide full reasoning behind their answers, broken down into logical steps. As a result we gain a unique window into their "thought process" and can better isolate where they make their errors. Analyzing the textual responses, we find that GPT-3.5 often ignores prior information and therefore lacks a Bayesian conceptual framework. GPT-4 generally understands that Bayes Rule is the relevant principle but makes algebraic errors in the process of computing the Bayesian posterior probability. However, GPT-4o shows remarkable improvement, transitioning from a "conceptual" to an "accurate" Bayesian approach.

Why should we care how "Bayesian" humans or machines are? Though the decision problem we study is a simple and somewhat abstract "textbook problem" we believe it is a reasonable approximation to and metaphor for a range of real world classification problems involving much higher stakes. One such example is the problem of making optimal *differential diagnoses* (DDx) that involve classifying which of several alternative diseases

[3]So as not to keep the reader in suspense, in figure 1 subject group 1 (blue line) are human subjects and subject group 2 (red line) are chatGPT 4 subjects.

or medical problems most likely caused by a set of observed symptoms in a patient. Recent studies (e.g. Goh et al. (2024) and McDuff et al. (2023)) have demonstrated that LLMs and GPTs can outperform human physicians in the quality and accuracy of their differential diagnoses. However unlike our problem, there is no objective measure of what the "correct" diagnosis is for most cases. These studies rely on the diagnoses of panels of expert physicians to score the choices made by GPT and less experienced interns, and use *ad hoc* scoring rules to evaluate the reasoning that lead subjects to choose a particular diagnosis, but without a clear notion of the cost or harm from a misdiagnosis. In comparison, we have an objective way of evaluating loss and classifying "easy" and "hard" problems to identify where subjects make their most costly mistakes.

Our paper contributes to two strands of literature. First, we build on and extend the literature of testing whether human behaviors are consistent with Bayes' rule (El-Gamal and Grether, 1995; El-Gamal and Grether, 1999; Holt and Smith, 2009). We formally show that applying Bayes' Rule is optimal in these experiments and develop a novel performance measure superior to widely used accuracy. To infer subjects' posterior subjective beliefs, we propose a new structural logit model, which we show provides a better fit of human behaviors than previously ones in the literature.

Second, we contribute to the growing literature understanding LLM behavior using experimental methods and comparing it with human behavior in a variety of contexts,[4] among them Chen et al. (2023) and Mei et al. (2024) are closest to us. Mei et al. (2024) evaluates the behavioral and personality traits of AI, including trust, fairness, risk-aversion, altruism, and cooperation and find that *"Their behaviors fall within the distribution of behaviors of humans and exhibit patterns consistent with learning"*. Chen et al. (2023) compares chatGPT and a corresponding sample of human subjects in their ability to make rational choices in four different domains involving risk, intertemporal choices, food choices and social choices and concludes that *"GPT's decisions are largely rational in each domain and demonstrate higher rationality score than those of human subjects in a parallel experiment and in the literature."* We instead evaluate ChatGPT's

[4]The literature finds that LLMs excel at textual tasks, for example, divergent thinking tasks (Hubert et al., 2024), ophthalmology exams ((Yan et al., 2024); (Taloni et al., 2023)), essay writing (Herbold et al., 2023), and uniform bar exams (Martínez, 2024). The literature also reports that LLMs can fall short of humans, at least in certain aspects, in some non-textual tasks, including complex mathematical proofs and computation (Frieder et al., 2023), analyzing healthcare data (Li et al., 2024), physics coding (Yeadon et al., 2024) and medical board exams (Katz et al., 2024). For additional studies in marketing and finance, see Goli and Singh, 2024, and Zhao et al. (2024) and the references there.

Bayesian rationality—a cornerstone of decision theory with broad applications in domains such as disease diagnosis, pricing strategies, financial investment, and beyond. We document a rapid improvement in decision efficiency, moving from varied noisy decision-makers to more uniform, near-Bayesian ones in a few generations. Our estimation of the structural model and novel analysis of textual responses offer deeper insights into how this increase in decision efficiency occurs.

The remainder of this paper is organized as follows. Section 2 reviews the relevant background on statistical decision theory that provides a simple inefficiency index that we use to summarize the degree of irrationality/suboptimality of both human and AI subjects relative to the optimal decisions given by Bayes Rules. Section 3 introduces a structural econometric model of subject responses that we use to infer subject beliefs and summarize their behavior in these experiments. In section 4 we summarize the findings of our reanalysis of human subject data reported in the studies by El-Gamal and Grether (1995), El-Gamal and Grether (1999) and Holt and Smith (2009). In section 5, we use the same experimental designs but replace the human subjects with AI subjects from various versions of currently commercially available GPT software, including ChatGPT. In Section 6, we examine the textual responses of GPT subjects to identify where mistakes occur in their reasoning process. Section 7 provides some overall conclusions and discuss the larger significance of our results including revised forecasts of the imminent demise of *homo sapiens.*

2 Statistical Decision Theory Background

In this section we review some basic concepts of statistical decision theory that provide a well defined metric for characterizing the degree of irrationality/inefficiency in the responses of both human and AI subjects. The human subject data we reanalyze in section 4 were gathered from four separate experiments: 1) 257 student subjects from four different universities in California reported in El-Gamal and Grether (1995), 2) 79 student subjects at the University of Wisconsin reported in El-Gamal and Grether (1999), 3) 22 subjects at the University of Virginia, and 4) 24 subjects who participated in web-based experiments, both reported in Holt and Smith (2009).

All of the experiments analyzed in this paper can be classified as "binomial designs"

where subjects are presented with two bingo cages, labeled A and B, each containing the same known number of balls of two types (i.e. balls labeled N or G in experiments 1 and 2, light and dark marbles in experiment 3, or red and blue balls in experiment 4). A credible random mechanism was used to select one of the two cages (e.g. selecting one of the cages based on a toss of a die or a random number generator) though subjects were not shown which cage was selected. A random sample of D balls with replacement was drawn from the selected cage and shown to the subjects.

In experiments 1 and 2 subjects were asked to choose the cage they believed was most likely to have been used to draw the sample. In experiments 3 and 4 subjects were asked to report the probability that cage A was the one from which the sample was drawn using an incentive-compatible procedure introduced by Becker et al. (1964) known as the *BDM mechanism* which involves a second stage lottery whose payoff depends on the probability the subject reports. This lottery is designed so that the payoff maximizing report is the subjective posterior probability of cage A. We will describe the BDM mechanism in more detail in section 4.3 below. In experiments 1 a subset of subjects were paid a $10 bonus if they selected the actual cage used to draw the random sample of balls for a randomly selected trial out of the total trials they participated in. In experiment 2 all subjects received a $20 bonus for each correct response in 3 randomly selected trials.

It is important to note that in all experiments incentive payments were made *after* all trials were completed. Beyond an initial description of the bingo cage setup and a single demonstration of how it works at the start of the experiment, *none of the subjects received any feedback on whether they had selected the correct cage after each trial in the experiment.* This was evidently an intentional feature of the experimental design, to reduce the possibilities of non-stationarity in subjects' decision rules during the experiment due to "learning-by-doing" that is enhanced by real-time feedback.[5]

The problem of selecting the cage from which the observed sample was most likely to have been drawn is an elementary *statistical decision problem* whose optimal solution is given by Bayes Rule. In the next section we summarize the relevant statistical decision theory that provides the conceptual framework for our comparison of human and AI

[5]We tested for the presence of learning by doing effects simply due to repeated participation even without feedback by comparing performance on the first third of trials with the last third. We find small learning by doing effects even for the non-incentivized subjects, even in the absence of sequential feedback about whether they had selected the correct cage after each trial. However the effect is sufficiently small that we ignore it in the subsequent analysis in this paper.

behavior. Then we introduce a *structural logit model* of subject decision making that we use to analyze human and AI responses and compare their degree of suboptimality using statistical decision theory.

2.1 Bayes Rule, Decision Rules, Win and Loss Functions

First we introduce the notation to more precisely describe these *statistical experiments.* Let d denote the number of balls in the sample of D balls that have a designated type (i.e. balls marked N in experiments 1 and 2, light balls in experiment 3 or red balls in experiment 4). Though d is a sufficient statistic for the full random sample, subjects were shown the full sample outcomes.[6] Let p_A and p_B be the probabilities of selecting the designated type of ball from each cage. The probabilities equal the fractions of the total number of balls in each cage of the designated type. Let $f(d|p_A, D)$ and $f(d|p_B, D)$ be the probabilities of observing d balls of the designated type in the random sample of D balls for cages A and B. These are binomial distributions with parameters (p_A, D) and (p_B, D), respectively. Finally, let $\pi \in (0, 1)$ denote the credible *objective prior probability* that cage A was selected to draw the random sample of D balls.

The behavior of subjects in the experiments can be summarized by a *decision rule* which is a function $\delta(d, \pi, p_A, p_B, D)$ mapping the information provided to subjects in the experiments into a choice of cage A or B. Following El-Gamal and Grether (1995) we do not assume all subjects use the same decision rule, and our analysis will attempt to identify different *types* of subjects who use similar decision rules, using statistical methods for discovering unobserved heterogeneity including the *Estimation-Classification* (EC) algorithm introduced by El-Gamal and Grether (1995) as well as a finite mixture model approach introduced by Heckman and Singer (1984).

Our analysis of human and AI subject data also allows for probabilistic decision rules (i.e. "mixed strategies") as well as pure strategies that appear probabilistic to the experimenter because the subject's choice depends on additional information or stochastic psychological "decision errors" ν that are observed only by the subject and not by the experimenter. This results in a decision rule of the form $\delta(d, \pi, p_A, p_B, D, \nu)$. For this reason we define the decision rule for a subject as a conditional probability of selecting

[6] In experiments 1 to 3 subjects were shown the results of each draw sequentially, but in experiment 4 the results of the draws were presented simultaneously "to mitigate the tendency of subjects to overweight recent observations" (p. 129).

cage A that depends on the public information (d, π, p_A, p_B, D).

Definition D1. Decision Rule: *Any conditional probability $P(A|d, \pi, p_A, p_B, D)$ of selecting cage A as a function of the publicly observable information (d, π, p_A, p_B) as well as other information or influences ε that are privately observed/experienced by the subject. If ν has CDF $F(\nu)$ that is independent of the public signals, then P has the representation*

$$\begin{aligned} P(A|d, \pi, p_A, p_B, D) &= Pr\{A = \delta(d, \pi, p_A, p_B, D, \nu)\}, \\ &= \int_{\varepsilon} I\{A = \delta(d, \pi, p_A, p_B, D, \nu)\} F(\nu). \end{aligned} \tag{1}$$

Note that P is also referred to as a *conditional choice probability* (CCP) and it does *not* represent the subjective beliefs about the probability that the sample came from cage A. In fact a decision rule δ may have no connection to any well defined subjective beliefs. For example a variety of machine learning algorithms such as support vector machines or neural networks can be trained to have nearly optimal decision rules but they do not require or depend on well defined subjective posterior beliefs about A.

However humans, as well as chatGPT, are able to report subjective posterior probabilities. Thus, we would like a model where the decision rule depends, at least implicitly, on their subjective beliefs. In section 3 we introduce a parametric *structural logit model* of subject choices that does explicitly depend on subjective posterior beliefs, and which in principle can be used to uncover subjective beliefs from observed choices. With sufficient data on subject choices in many trials for different values of (d, π, p_A, p_B, D), it is possible to estimate P non-parametrically even in situations where it is not possible to identify subjective posterior beliefs. As we show, knowledge of P is sufficient for our key conclusions about the relative efficiency of human vs AI subjects.

Definition D2. Bayes Rule: *The conditional probability that cage A was selected given the information (d, π, p_A, p_B, D) given by*

$$\Pi(A|d, \pi, p_A, p_B, D) = \frac{\pi f(d|p_A, D)}{\pi f(d|p_A, D) + (1 - \pi) f(d|p_B, D)}. \tag{2}$$

Bayes Rule is often interpreted as an individual's *subjective posterior beliefs* since the prior belief π is treated as a subjective prior probability that can differ from subject to subject. However due to the setup of these experiments, π should be interpreted as an

objective prior probability that cage A is selected, and we assume that this information has been credibly communicated to subjects. Thus, it follows that the Bayesian posterior probability $\Pi(A|d, \pi, p_A, p_B, D)$ given in equation (2) is an *objective posterior belief* i.e. it is the true conditional probability that cage A is selected given (d, π, p_A, p_B, D).

Even though some of the subjects were not incentivized via a bonus payment for selecting the correct cage in a randomly selected trial, it seems reasonable to presume that both incentivized and non-incentivized subjects were trying to maximize the probability of selecting the correct cage. In other words, we presume that subjects in the experiment will want to use an *optimal decision rule* if it is not too difficult for them to calculate, or some approximation to it otherwise.

It is convenient to define two binary random variables, $\tilde{W}_P$ and $\tilde{L}_P$, implied by decision rule P by $\tilde{W}_P = 1$ if the subject selects the correct cage from which the sample was drawn, and 0 otherwise. Thus, $\tilde{W}_P$ is an indicator for a "win" i.e. a correct prediction or classification. $\tilde{L}_P$ is the indicator for a loss, i.e. an incorrect prediction. It follows that with probability 1 we have $1 = \tilde{W}_P + \tilde{L}_P$, and so we can define an optimal decision rule as one that maximizes the probability of a win or conversely one that minimizes the probability of a loss. Following the standard terminology from the literature on statistical decision theory, we define

Definition D3. Loss Function *The loss function is the conditional probability of a loss,*

$$\begin{aligned} L_P(d, \pi, p_A, p_B, D) &= E\{\tilde{L}_P|d, \pi, p_A, p_B, D\} \\ &= P(A|d, \pi, p_A, p_B, D)[1 - \Pi(A|d, \pi, p_A, p_B, D)] \\ &+ [1 - P(A|d, \pi, p_A, p_B, D)]\Pi(A|d, \pi, p_A, p_B, D). \end{aligned} \tag{3}$$

Definition D4. Win Function *The win function is the conditional probability of a win, i.e. selecting the correct cage,*

$$W_P(d, \pi, p_A, p_B, D) = E\{\tilde{W}_P|d, \pi, p_A, p_B, D\} = 1 - L_P(d, \pi, p_A, p_B, D). \tag{4}$$

An *optimal decision rule* P maximizes the probability of a win, or equivalently it minimizes the probability of a loss. Using equation (3) or (4) the optimal decision rule is the pure strategy (5) defined in terms of Bayes Rule in Lemma L1.

Lemma L1. *The optimal decision rule for a statistical experiment with a binomial design can be defined in terms of Bayes Rule by*

$$\delta^*(d,\pi,p_A,p_B,D) = \begin{cases} A & \textit{if } \Pi(A|d,\pi,p_A,p_B,D) \geq 1/2 \\ B & \textit{otherwise.} \end{cases} \tag{5}$$

The win function W_P conditions on the full information set (d,π,p_A,p_B,D). Of course, d varies stochastically over trials for each subject, but the other 4 variables serve as experimental controls. For example in the California experiments the prior π varied across trials taking on three possible values, $\pi \in \{1/3, 1/2, 2/3\}$ but the remaining variables were fixed at $D = 6$, $p_A = 2/3$ and $p_B = 1/2$. In the Wisconsin experiments π also varied over trials but the remaining variables (D, p_A, p_B) only changed over successive rounds (groups of trials) taking values $(6, 2/3, 1/2)$ in round 1 and $(7, .4, .6)$ in round 2.

In our comparisons of the performance of human vs AI decision makers, it is convenient to have a single overall scalar summary measure of *decision efficiency* which we define as the ratio of the subject's expected win probability to the optimal expected win probability implied by Bayes Rule, where we compute expectations over the empirical distributions for the values of the experimental controls in the experiments. For example, we can define a *ex ante* or unconditional expected loss by first taking expectations over the unconditional distribution over the realized values of d given (π, p_A, p_B, D) by

$$\begin{aligned} W_P(\pi,p_A,p_B,D) &= E\{\tilde{W}_P|\pi,p_A,p_B,D\} \\ &= \sum_{d=0}^{D} W_P(d,\pi,p_A,p_B,D)[f(d|p_A,D)\pi + f(d|p_B,D)(1-\pi)]. \end{aligned} \tag{6}$$

If $F(\pi,p_A,p_B,D)$ is the empirical distribution of the experimental control variables in all trials of the experiment, the overall expected loss for a subject with decision rule P in this experiment is given by

$$W_P = E\{\tilde{W}_P\} = \int_\pi \int_{p_A} \int_{p_B} \int_D W_P(\pi,p_A,p_B,D) dF(\pi,p_A,p_B,D). \tag{7}$$

We will use W_P as a single summary statistic for the overall performance of decision rule P using our econometric estimates of P from our structural logit model of subject choice behavior discussed below. Similarly, we can define a single summary statistic for the

optimal win probability in the same experimental design under the optimal decision rule implied by Bayes Rule, W_{δ^*}. Thus our overall scalar efficiency metric is given by ω_P equal to the ratio of the subject's expected win probability to the optimal win probability of a perfect Bayesian decision maker, $\omega_P = W_P/W_{\delta^*}$. Clearly we have $0 \leq \omega_P \leq 1$.

3 Structural Econometric Model of Subject Responses

This section introduces an econometric model of subject decision making that we refer to as a *structural logit model.* It is related to and subsumes as special cases the "probability weighting" model used by Holt and Smith (2009) to analyze reported posterior and a "structural probit" model introduced by El-Gamal and Grether (1999). It also includes the optimal Bayesian decision rule as a special case as well. The structural logit model also has an interpretation as a two layer neural network where the first input layer uses "transformed inputs" equal to the log-likelihood ratio and the log posterior odds ratio and the second output layer uses the subjective posterior probability output from the first layer as its input and includes it in a logistic "squashing function" that is a monotonic function of the difference between the subjective posterior and 1/2. A key advantage of the structural logit model is that it enables us to recover estimates of subjective posterior beliefs even when subjects only make binary choices of which cage they believe is more likely to have generated the observed sample d.

We compare the structural logit model to an alternative structural model introduced by El-Gamal and Grether (1995) that assumes that subjects use *cutoff rules* to make their decisions. Their cutoff rule model does not require subjects to have subjective posterior beliefs, and thus bypasses trying to estimate them. They estimated this model using data from their California experiment cage A where $D = 6$ and $p_A = 2/3$ and $p_B = 1/2$. Since higher realized values for d provide stronger evidence in favor of cage A, a plausible cutoff rule involves choosing cage A if $d > c_\pi$ where c_π is a prior-specific threshold. In their California design there were three possible priors, $\pi \in \{1/3, 1/2, 2/3\}$ so there are 512 possible cutoff rules $(c_{1/3}, c_{1/2}, c_{2/3})$ that subjects could use. For example, Bayes Rule corresponds to the cutoff rule $c = (c_{1/3}, c_{1/2}, c_{2/3}) = (4, 3, 2)$. However the cutoff rule model is *statistically degenerate* i.e. for most subjects observed over T successive trials, none of the cutoff rules will be able to perfectly predict the subject's choices. To deal with

this and avoid a "zero likelihood problem" El-Gamal and Grether (1995) introduced an additional error rate parameter $\sigma \in (0,1)$ which has the interpretation as the probability that the subject guesses one of the cages at random rather than using the cutoff rule. This results in a non-degenerate choice probability (decision rule) given by

$$P(A|d,\pi,p_A,p_B,D) = \begin{cases} 1-\frac{\sigma}{2} & \text{if } d > c_\pi \\ \frac{\sigma}{2} & \text{otherwise.} \end{cases} \tag{8}$$

Using the choice probability $P(A|d,\pi,p_A,p_B,D)$ in equation (8), El-Gamal and Grether (1995) were able to write a likelihood function of the sequence of choices made by each subject in the T trials in the experiment, and use it to estimate the four unknown parameters of their model, $\theta = (c_{1/3}, c_{1/2}, c_{2/3}, \sigma)$ by maximum likelihood.

Though the model is innovative and interpretable, it has several drawbacks that lead us to abandon it in favor of the structural logit model we introduce below. First, the cutoff rule parameters are not "structural" i.e. they are not invariant to changes in the experimental controls (π, p_A, p_B, D). For example, the cutoffs c_π are only estimated for the three possible values of π used in their experiment. Ideally, a structural model should depend on parameters that are *policy invariant* and do not change when the experimental design controlling the "policy" or "environment" is changed. Second, the nature of subject errors implied by their model is not entirely plausible: the probability that a subject randomly guesses, σ, does not depend on the strength of the evidence the subject observes. For example, for values (d,π,p_A,p_B,D) we should not expect to see much guessing in the "easy cases" where the Bayesian posterior probability is close to 0 or 1. Random guesses should be prevalent mostly in ambiguous cases where the Bayesian posterior is close to 1/2.

We now introduce a *structural logit model* that avoids these limitations. It implies a parametric family of stochastic decision rules $\delta(d,\pi,p_A,p_B,D,\nu,\varepsilon)$ and stochastic subjective posterior beliefs $\Pi_s(A|d,\pi,p_A,p_B,D,\nu)$ that includes the optimal Bayesian decision rule $\delta^*(d,\pi,p_A,p_B,D)$ as a special case. The structural logit model depends on 5 unknown parameters $\theta = (\beta_0,\beta_1,\beta_3,\sigma,\eta)$ where the first three parameters are allowed to take any real value and determine the subject's subjective posterior beliefs. The last parameters η and σ are restricted to be positive and determine the scaling/level of idiosyncratic "noise" affecting the subject's posterior beliefs via ν, and final choice via ε.

Consider first the subject's subjective posterior beliefs $\Pi_s(A|d, \pi, p_A, p_B, D, \nu)$, where ν is a random variable that reflects stochastic shocks and "calculational errors" that subjects experience trying to compute their subjective posterior probability. We presume that subjects attempt to transform their information (d, π, p_A, p_B, D) into $\text{LPR}(\pi)$ and $\text{LLR}(d, p_A, p_B, D)$ where $\text{LPR}(\pi)$ is the log posterior odds ratio and $\text{LLR}(d, p_A, p_B, D)$ is the log-likelihood ratio given by

$$\begin{aligned} \text{LPR}(\pi) &= \log(\pi/(1-\pi)) \\ \text{LLR}(d, p_A, p_B, D) &= \log(f(d|p_A, D)/f(d|p_B, D)). \end{aligned} \tag{9}$$

However we also assume that both human and AI subjects can make algebraic errors trying to evaluate the quantities $\text{LPR}(\pi)$ and $\text{LLR}(d, p_A, p_B, D)$. Let the scalar random variable ν equal the sum of these errors, so the log-posterior odds ratio that the subject actually perceives and reports (or makes their choice on), $\Pi(A)$, is given by

$$\log\left(\Pi_s(A)/(1-\Pi_s(A))\right) = \beta_0 + \beta_1 \text{LLR}(d, p_A, p_B, D) + \beta_2 \text{LPR}(\pi) + \nu. \tag{10}$$

In our empirical analysis below we assume $\nu \sim N(0, \eta^2)$. Solving equation (10) for $\Pi_s(A|d, \pi, p_A, p_B, D, \nu)$ results in the following logistic specification given by

$$\Pi_s(A|d, \pi, p_A, p_B, D, \nu) = \frac{\exp\{\beta_0 + \beta_1 \text{LLR}(d, p_A, p_B, D) + \beta_2 \text{LPR}(\pi) + \nu\}}{1 + \exp\{\beta_0 + \beta_1 \text{LLR}(d, p_A, p_B, D) + \beta_2 \text{LPR}(\pi) + \nu\}}. \tag{11}$$

Notice that the true Bayesian posterior $\Pi(A|d, \pi, p_A, p_B, D)$ given in equation (2) is a special case of (11) when $\beta = (0, 1, 1)$ and $\nu = 0$. For other values of β the subjective posterior can capture a number of well known biases observed in past studies, including an outright bias for cage A or B if $\beta_0 \neq 0$ as well as *overconfidence* and *underconfidence* about the posterior probability of cage A, *base rate bias* ($\beta_2 < \beta_1$) resulting in behavior consistent with the representativeness heuristic (i.e. excessive weight on the data via LLR relative to the prior via $\text{LPR}(\pi)$), as well as conservatism ($\beta_2 > \beta_1$, i.e. putting excessive weight on prior information relative to sample information).

Next we model the subject's binary choice of cage A or B, which we assume depends on whether their perception of the expected reward from choosing cage A exceeds the expected reward from choosing cage B. Suppose the subject receives a reward R if they

select the correct cage and 0 otherwise. In experiments where subjects were not paid for making a correct choice, R can be viewed as a "psychological reward" the subject expects from making a correct choice. However we assume that the subject's expected reward for choosing either cage can also be affected by unobserved idiosyncratic preference shocks $\varepsilon = (\varepsilon(A), \varepsilon(B))$ that we assume are distributed independently of ν. In our empirical analysis below we assume that ε has a bivariate Type 1 extreme value distribution with location parameter normalized to 0 and a common scale parameter σ. Normally we would expect the subject to select cage A if $\Pi_s(A|d, \pi, p_A, p_B, D, \nu) > 1/2$ and cage B otherwise. However we allow the subject's report to be affected by the additional preference shocks ε to capture behavior such as simply guessing one of the cages, or some other psychological factors that may cause a subject to report a cage that may not have the higher posterior probability. This specification implies the following decision rule for the subject

$$\delta(d, \pi, p_A, p_B, D, \nu, \varepsilon) = \begin{cases} A & \text{if} \quad R\Pi_s(A) + \sigma\varepsilon(A) \geq R\Pi_s(B) + \sigma\varepsilon(B) \\ B & \text{otherwise.} \end{cases} \tag{12}$$

As is well known from the discrete choice literature (see e.g. McFadden (1974)) the probability that the subject chooses cage A is given by the binomial logit formula

$$\begin{aligned} P(A|d, \pi, p_A, p_B, D, \nu) &= \Pr\{\delta(d, \pi, p_A, p_B, D, \nu, \epsilon) = A|d, \pi, p_A, p_B, D, \nu\} \\ &= \frac{\exp\{R\Pi_s(A|d, \pi, p_A, p_B, D, \nu)/\sigma\}}{\exp\{R\Pi_s(A|d, \pi, p_A, p_B, D, \nu)/\sigma\} + \exp\{R\Pi_s(B|d, \pi, p_A, p_B, D, \nu)/\sigma\}} \\ &= \frac{1}{1 + \exp\{R[1 - 2\Pi_s(A|d, \pi, p_A, p_B, D, \nu)]/\sigma\}}. \end{aligned} \tag{13}$$

It follows that when $\Pi_s(A|d, \pi, p_A, p_B, D, \nu) = 1/2$ the subject is indifferent between choosing cage A or B and the noise terms $(\varepsilon(A), \varepsilon(B))$ determine the subject's choice, so $P(A|d, \pi, p_A, p_B, D, \nu)$ is also equal to 1/2. However as $\Pi_s(A|d, \pi, p_A, p_B, D, \nu)$ approaches 0 or 1, the "strength of the evidence" reduces the role of the idiosyncratic shocks ε on the subject's choice. Thus, $P(A|d, \pi, p_A, p_B, D, \nu)$ increases to 1 when R is sufficiently large or σ is sufficiently small and $P_s(A|d, \pi, p_A, p_B, D, \nu) > 1/2$, and conversely $P(A|d, \pi, p_A, p_B, D, \nu) \to 0$ when $P_s(A|d, \pi, p_A, p_B, D, \nu) < 1/2$ as $R/\sigma \to \infty$.

Notice that that the structural logit model results in a "mixed strategy" for the choice of bingo cage due to the effect of the idiosyncratic shocks ε. However as $\sigma \to 0$ it converges

to a pure strategy that chooses cage A with probability 1 when $\Pi_s(A|d,\pi,p_A,p_B,D,\nu) > 1/2$ and cage B when the subjective posterior is strictly less than 1/2. Further if $\beta = (0,1,1)$ and $\eta = 0$ (so that subjects do not make random errors calculating LPR and LLR), then their subjective posterior belief coincides with the Bayesian posterior belief and the structural logit model encompasses the optimal pure strategy Bayesian decision rule $\delta^*(A|d,\pi,p_A,p_B,D)$ given in equation (5) as a special case.[7]

In experiments 1 and 2 subjects only report which cage is more likely, so the unobserved "calculational error" ν must be integrated out to produce a CCP that can be used for estimation of the model. When $\nu \sim N(0,\eta^2)$ the CCP is a mixed logit given by

$$P(A|d,\pi,p_A,p_B,D) = \int \frac{1}{1+\exp\{R[1-2\Pi_s(A|d,\pi,p_A,p_B,D,\eta\nu)]/\sigma\}}\phi(\nu)\,d\nu, \qquad (14)$$

where $\phi(\nu)$ is the standard normal density.

The structural logit model can also be interpreted as a two layer feedforward neural network (or 3 layer in the mixed logit case) that uses transformed inputs $\mathrm{LPR}(\pi)$ and $\mathrm{LLR}(d,p_A,p_B,D)$ and produces a single output: the probability of choosing cage A. Maximum likelihood of the structural logit model can be interpreted as training the neural network to behave like a human being. The ability of the structural logit model to fit a wide range of subject behaviors can be ascribed to the flexibility afforded by using a parsimoniously parameterized neural network to predict subject behavior.[8]

Regardless of how it is interpreted, the structural logit implies a CCP that is a function of the unknown parameter vector $\theta = (\beta,\eta,\sigma)$ that can be estimated by maximum likelihood. We use a panel likelihood function since each subject s in the experiment participates in a total of T_s independent trials, so we observe a sequence of binary choices d_{ts} and corresponding controls (π_{ts}, D_{ts}) for each subject s over trials $t = 1,\ldots,T_s$ assuming

[7] El-Gamal and Grether (1999) introduced a three parameter *structural probit model* which is a decision rule of the form $P(A|d,\pi,p_A,p_B,D,\beta) = \Phi(\beta_0 + \beta_1\mathrm{LLR}(d,p_A,p_B,D) + \beta_2\mathrm{LPR}(\pi))$ where Φ is the standard normal CDF. The optimal Bayesian decision rule $\delta^*(d,\pi,p_A,p_B,D)$ is not nested within this class of models, though it can be approximated arbitrarily well in the limit when $\beta_0 = 0$ and $\beta_1 = \beta_2 = \beta > 0$ and $\beta \to \infty$.

[8] It is not necessary to pre-transform the inputs (d,π,p_A,p_B,D) into $(\mathrm{LPR}(\pi),\mathrm{LLR}(d,p_A,p_B,D))$: additional layers can be added to the neural network so that the inputs to the deeper neural network can entered without any pre-transformation. Then the initial layers of this deeper neural network can be viewed as producing approximations to the transformed inputs $(\mathrm{LPR}(\pi),\mathrm{LLR}(d,p_A,p_B,D))$ that then feed into the two layer neural net that used the transformed inputs to compute a subjective posterior probability and the top layer producing an output equal to the conditional probability of selecting cage A. These deeper networks require far more parameters, but do not result in substantially better predictions of subject behavior than our parsimonious 4 parameter two layer neural network specification. Indeed we can "train" our 4 parameter neural network specification to behave nearly identically to a perfect Bayesian decision maker using training samples with only a few dozen observations.

(p_A, p_B) remain fixed across trials. Let y_{ts} be a binary indicator of the choice of subject s in trial t: $y_{ts} = 1$ if the subject chose A and $y_{ts} = 0$ otherwise. The likelihood $L(\theta)$ is given by

$$L(\theta) = \prod_{s=1}^{S} \prod_{t=1}^{T_s} P(A|d_{ts}, \pi_{ts}, p_A, p_B, D_{ts}, \theta)^{y_{ts}} [1 - P(A|d_{ts}, \pi_{ts}, p_A, p_B, D_{ts}, \theta)]^{1-y_{ts}}. \quad (15)$$

3.1 Identification of Beliefs

Both of the unobserved private shocks ε and ν imply idiosyncratic responses by subjects, so even the same subject can provide two different choices to the same information (d, π, p_A, p_B, D). Since these shocks have similar effects on a subject's choice, it is not clear whether it is possible separately identify the independent effect of each type of shock using data from experiments 1 and 2 where subjects provide only binary responses. For these experiments we estimated two different restricted versions of the mixed structural logit specification (14): 1) a 3 parameter *structural probit* that restricts $\sigma = 0$ and normalize $\eta = 1$, 2) a 4 parameter structural logit model where set $\eta = 0$ but estimate the scale parameter σ of the extreme value distribution of the preference shocks ε.

In experiments 3 and 4 subjects reported their subjective posterior probabilities but did not make binary choices. So we assume that subjects would select cage A if and only if their reported subjective posterior probability exceeds 1/2, which is equivalent to restricting $\sigma = 0$. However we estimate the standard deviation parameter η of the "calculational errors" in their reported subjective posteriors. It is not hard to see from equation (10) that the belief parameters β and η are parametrically identified given sufficient subject data and sufficient variation in the experimental controls (π, p_A, p_B, D) to avoid multicollinearity in the covariates $\text{LLR}(d, p_A, p_B, D)$ and $\text{LPR}(\pi)$.

Identification of the parameters is more challenging in the case where subjects only report binary choices of cage A or B, even under the restriction that $\eta = 0$. First, we observe that it is impossible to separately identify the reward R and the error or noise parameter σ since it is obvious from formula (13) that these parameters only appear together as a "signal to noise ratio" R/σ in the decision rule which in turn is used to form the likelihood for the data. Thus, we assume that the reward R from making a correct decision is known (e.g. it equals the \$10 bonus for picking the correct cage) and we normalize the payoff to $R = 1$ and estimate only σ subject to this normalization.

Thus, one must keep in mind that a high value of σ may not necessarily indicate a high level of random noise affecting the subject's decision making: it could also indicate that the subject's evaluation of the reward from a correct decision, R, is sufficiently low. In our empirical reanalysis of El-Gamal and Grether (1995) the structural logit model estimates of σ are lower in the incentivized experiments where subjects received a \$10 bonus if they correctly selected the cage in a randomly selected trial than in the "no pay" experiments. The lower estimated σ may reflect the higher reward and not necessarily that the scale of the random preference shocks is lower among the incentivized subjects.

An important question is whether the estimated structural logit model provides accurate estimates of subjects' posterior beliefs, something we do not directly observe in the El-Gamal and Grether experiments. Ours is an exercise in *revealed beliefs* where we attempt to uncover subjective posterior beliefs from observations of subject's binary choices. In fact in the case where $\sigma = 0$, we already have an identification problem in that there are a continuum of non-Bayesian posterior beliefs that are consistent with the optimal Bayesian decision rule $\delta^*(d, \pi, p_A, p_B, D)$. To see this consider a family of beliefs indexed by a single parameter $\lambda > 0$ given by $\theta_\lambda = (\beta_\lambda, \sigma, \eta)$ where $\beta_\lambda = (0, \lambda, \lambda)$ and $\eta = 0$ and $\sigma = 0$. For any λ and any information (d, π, p_A, p_B, D) we have

$$\Pi(A|d, \pi, p_A, p_B, D) \leq 1/2 \iff \Pi_s(A|d, \pi, p_A, p_B, D, \theta_\lambda) \leq 1/2. \tag{16}$$

This equivalence implies that the implied decision rule for the subject given in equation (12) coincides with the optimal Bayesian decision rule $\delta^*(A|d, \pi, p_A, p_B, D)$ given in equation (5). Thus, even though the subject's posterior beliefs are not Bayesian, their decision rule is still optimal and achieves the same minimal loss as the one defined in terms of Bayes Rule. However we can show that the subject's posterior beliefs can be identified when $\sigma > 0$ if the experiment exposes the subject to priors that can be arbitrarily close to 0 or 1

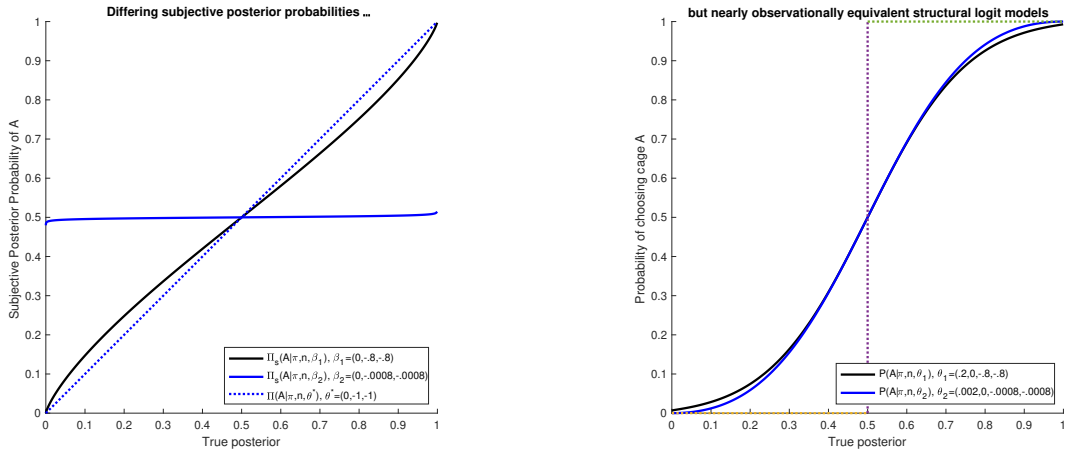
Lemma L2. Identification of subject beliefs when $\sigma > 0$ *Assume that* $\eta = 0$. *When* $\sigma > 0$, *all four parameters of the structural logit model are identified, so the subject's subjective beliefs can be identified from knowledge of their decision rule* $P(A|\pi, n)$, *where the latter is identifiable given sufficient experimental data on a subject's choices.*

The proof of Lemma L2 is in Appendix A. Even though Lemma L2 provides a theoret-

ical justification for the identification of the model when $\sigma > 0$, in practice it can be hard to distinguish the decision rule of a subject with Bayesian posterior beliefs where σ takes on relatively large values (i.e. a "noisy Bayesian") from a decision rule of a non-Bayesian who has a very small value of σ but whose β coefficients are also close to zero.

The weak identification of subjective posterior beliefs is illustrated in figure 2. The figure plots the CCPs for two very different values of the structural parameters: $\theta_1 = (0, .8, .8, .2, 0)$ and $\theta_2 = (0, .008, .008, .002, 0)$, i.e. $\theta_2 = \theta_1/100$. The left panel plots the subjective posterior beliefs for these two different "subjects" and we can see that they are very different, both from each other and from the true Bayesian posterior (the dashed 45 degree line). The right hand panel shows that the implied CCPs are nearly identical. This example does not contradict our identification result, Lemma L2, because we can see that $P(A|d, \pi, p_A, p_B, D, \theta_1) \neq P(A|d, \pi, p_A, p_B, D, \theta_2)$ but the differences are really only apparent at extreme values such as when π is close to 0 or 1. So even though the structural logit is "technically identified" as a theoretical matter, but in any practical sense it is only weakly identified.[9]

Figure 2: Example of weak identification of subjective posterior beliefs



[9]One way to understand the identification problem is to note that the structural probit model of El-Gamal and Grether (1999) discussed in footnote 7 must be subject to a location scale normalization on its normally distributed error term. As we discussed above in the structural logit model the extreme value scale parameter σ only appears as a ratio R/σ where R is the payoff for a correct classification. Even if we normalize $R = 1$ and Lemma L2 assures us that the structural logit CCP is not scale invariant in its 4 parameters θ, it is *approximately scale invariant* as shown in figure 2. However imposing an arbitrary normalization such as $\sigma = 1$ has behavioral implications, since it determines the probability of incorrect classifications at the extreme values for the prior probability of choosing cage A, i.e. $\pi = 0$ or $\pi = 1$. Indeed, our proof of identification used the knowledge of these classification error probabilities at $\pi = 0$ and $\pi = 1$ to identify σ. But in actual data sets we may not have sufficient data at $\pi = 0$ or $\pi = 1$ and in such cases the identification of σ and the entire parameter vector θ will be more tenuous.

As a result, the focus of the empirical work in this section will be on the *optimality of subjects' decision rules* (which are non-parametrically identified) not necessarily on whether their posterior beliefs are Bayesian. Condition (16) can be viewed as a sufficient condition for optimality of the decision rule of a non-Bayesian subject and it results in a test for a weaker form of Bayesian rationality: $H_o : \beta_0 = 0$ and $\beta_1 = \beta_2$. If this latter hypothesis is satisfied, then the subject will still be modeled as behaving as a "noisy Bayesian" even though their posterior beliefs are not Bayesian. In section 4.3 we return to the question of inferring subjective posterior beliefs by reanalyzing the experiments of Holt and Smith (2009) that directly elicited subjects' beliefs.

3.2 Accounting for Unobserved Subject Heterogeneity

The experimental data we received from El-Gamal and Grether (1995) and El-Gamal and Grether (1999) do not contain any covariates or observable characteristics of the subjects in the experiment beyond the particular school (out of 4 in California) where the experiment was run. They found that students at the more selective schools (e.g. UCLA) were more likely to be Bayesian and less likely to guess.

However there is undoubtedly additional heterogeneity in the decision rules that subjects are using, including different thought processes (and different subjective posterior beliefs) that affect their choices. One possible way to account for this additional *unobserved heterogeneity* is to use *fixed effects* where we estimate subject-specific parameter vectors. However with S subjects observed over T time periods, we have ST total observations but a total of $4S$ parameters in the structural logit model. The time dimension (i.e. number of trials) is rather limited (in the California experiments $T \leq 20$). Since both T and S are relatively small, the *incidental parameters problem* of Kiefer and Wolfowitz (1956) suggests that inferences from a fixed effects approach are unreliable.

Instead we use a *random effects* approach to inferring unobserved subject heterogeneity, where following Kiefer and Wolfowitz (1956) we posit a *distribution* $\mu(\theta)$ of preference parameters in the population and attempt to estimate it. Treating μ as an arbitrary element of the space of all distributions over θ results in an infinite dimensional "parameter space" and the estimation problem can be ill-posed unless some restrictions are imposed. Following Heckman and Singer (1984) we estimate a finite mixture approximation to μ by maximum likelihood, treating this mixture as a *sieve* (i.e. an expanding parametric

family that increases with sample size S and can eventually approximate any μ when S and the number of mixture components is sufficiently large).

Let K denote the number of unobserved types to be estimated, and $\theta = (\theta_1, \ldots, \theta_K)$ be the $4K \times 1$ vector of parameters of the "mixed structural logit model" and let $\lambda = (\lambda_1, \ldots, \lambda_K)$ be the corresponding $K \times 1$ vector of population probabilities of each of the K types. Then the mixed logit likelihood function, $L(\theta, \lambda)$ is given by

$$L(\theta, \lambda) = \prod_{s=1}^{S} \sum_{k=1}^{K} \lambda_k L_s(\theta_k), \tag{17}$$

where $L_s(\theta_k)$ is the likelihood function for subject s evaluated at θ_k. We estimate a sequence of such models starting with $K = 1$ and increasing the number of types K until a log-likelihood ratio test is unable to reject a model with K types in favor of a model with $K + 1$ types, or alternatively, picking the value of K for which the Akaike Information Criterion (AIC) equals $2(N_K - L(\hat{\theta}, \hat{\lambda}))$ where N_K is the total number of parameters in the K-type model, is minimized.

We also estimate a closely related random effects approach called the *Estimation-Classification* (EC) algorithm by El-Gamal and Grether (1995). This algorithm also maximizes a likelihood function but instead of computing a mixture over types for each subject as in equation (17) the EC algorithm *assigns each subject in the sample their most likely type.* That is, the EC algorithm is the $4K \times 1$ vector $\theta = (\theta_1, \ldots, \theta_K)$ together with the $S \times 1$ vector of maximum likelihood subject classifications $\kappa = (k_1, \ldots, k_S)$ where k_s denotes the most likely type for subject s. That is, we can write the EC estimator $(\hat{\theta}, \hat{\kappa})$ as the pair (θ, κ) that maximizes the following likelihood

$$L(\theta, \kappa) = \prod_{s=1}^{S} \max_{1 \le k \le K} L_s(\theta_k), \tag{18}$$

where $L_s(\theta_k)$ is also the likelihood function for subject s evaluated at θ_k. The Heckman-Singer finite mixture estimator can be conceptualized as a "mixed strategy" since it treats each subject as a mixture over K possible types, whereas the EC estimator can be regarded as a "pure strategy" since instead of using a mixture over types, it assigns each subject their most likely type. The EC algorithm results in the following implied

mixture probabilities,

$$\hat{\lambda}_k = \frac{1}{S}\sum_{s=1}^{S} I\{\hat{k}_s = k\}, \tag{19}$$

where $\hat{k}_s$ is the type that maximizes the likelihood for subject s. Thus $\hat{\lambda}_k$ is simply the fraction of subjects whose most likely type is k. Similar to the finite mixture estimator, we use the same sequential procedure to identify the number of types K by starting with $K = 1$ and sequentially increasing K until a likelihood ratio test fails to reject the hypothesis that a model with K types fits significantly better than a model with $K + 1$ types. Alternatively we can pick the value of K with the smallest AIC.

4 Optimality and revealed beliefs of human subjects

4.1 Reanalysis of California Experiments

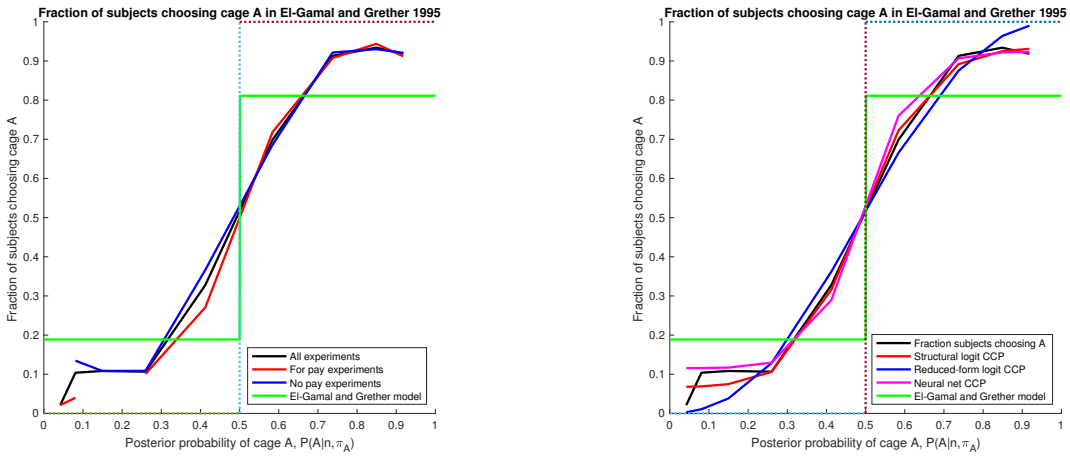
We estimated the El-Gamal and Grether threshold model and the structural logit model for 221 of the 247 subjects reported in El-Gamal and Grether (1995).[10] The results are shown graphically in figure 3 which compare predicted subject choices for several different models and subsamples. In both panels, we plot of the fraction of subjects choosing cage A (y axis) as a function of the true Bayesian posterior probability of cage A (x axis). The black lines in both panels are the actual fraction of the 221 subjects who chose cage A in the different trials of the experiment where the values of the "treatment variables" (π, n) are binned so we can plot results on a two dimensional graph with the Bayesian posterior probabilities, $\Pi(A|\pi, n)$, on the x-axis. Similar to figure 1 the dashed blue line represents the optimal decision rule of a perfect Bayesian decision maker.

The left panel illustrates the effect of the incentive payments on subject behavior (blue curve for the no-pay subjects, red for the subjects who were paid) and it is evident that it has negligible effect on overall behavior.[11] The maximum likelihood predictions from El-Gamal and Grether's model of subject choice are the green curves in both panels. We see that this model fails to fit the data well, particularly for the "easy cases", i.e. (π, n) values where the Bayesian posterior probability is near 0 or 1. But it also misses near

[10] Due to a corrupted data file we were unable to include 26 subjects from Pomona Community College under the incentivized (i.e. for pay) design.

[11] The average decision efficiency for the 90 subjects in the incentivized trials was 93.5% (std error 0.5%) which is not significantly higher than the 92.3% efficiency of the 132 subjects in the non-incentivized trials (std error 1.5%). We also separately analyzed data from the first and last third of the trials see if there were any substantial "learning by doing" or "experience effects" and these were also negligible.

Figure 3: Comparison of subject behavior and models in the California experiments



the "'hard cases" where the true posterior is near 1/2. This pattern of prediction errors follows from their assumption about subject behavior already discussed, namely that with probability $\sigma = .38$ subjects randomly guess a cage otherwise with probability $1-\sigma = .62$ they follow Bayes Rule. This implies a discontinuous jump in the predicted probability of selecting cage A right at $\Pi(A|\pi, n) = 1/2$ since at that point the 62% of subjects who are choosing according to Bayes Rule jump from choosing cage B to choosing cage A.

The right hand panel of figure 3 plots the predictions from the structural logit model (red curve) as well as several "reduced form" models: 1) a binary logit model with 3 parameters (for a constant, coefficient of π and coefficient of n), and 2) a 5 parameter two layer neural network that includes an additional bias parameter in the upper output layer. We can see visually that the structural logit model fits the data significantly better than the El-Gamal and Grether model even though both models have 4 parameters. The El-Gamal and Grether model restricts 3 of the parameters, the cutoffs c_π, to a finite grid of integers, which allows far less flexibility in fitting the data compared to the 4 continuous parameters of the structural logit. The structural logit also outperforms the 3 parameter reduced form logit (which can be regarded as a single layer feedforward neural network), but produces approximately the same predictions as a 5 parameter neural network specification. The structural logit model can be viewed as a restricted 4 parameter version of the 5 parameter neural network where the bias term in the output layer is restricted to be -1/2 times the value of the input weight parameter, which is $1/\sigma$ in the notation of the structural logit model. Intuitively, the structural logit model imposes

a harmless restriction that subjects compare their subjective posterior $\Pi_s(A|\pi, n)$ to the value 1/2 as the threshold for choosing cage A, and in particular, the restriction implies that subjects are equally likely to choose cage A or B when $\Pi_s(A|\pi, n) = 1/2$.

Table 1: Log-likelihood values for alternative models of subject choices

Model	**Number of parameters**	**Log-likelihood**	**AIC**
El-Gamal/Grether discrete cutoff rule	4	-1952	3912
Structural probit	3	-1847	3700
Reduced-form logit	3	-1821	3648
Noisy Bayesian	1	-1801	3604
Structural logit	4	-1773	3554
Neural network	5	-1772	3554

Table 1 summarizes the fit of the various models of subject behavior. The final column of the table reports the Akaike Information Criterion (AIC) used for model selection and defined as $2(k - LL)$ where k is the number of parameters in the model and LL is the maximized value of the log-likelihood function for that model. Though the 4 parameter El-Gamal and Grether model is not nested as a special case of the 4 parameter structural logit model, using the non-nested likelihood-based specification test of Vuong (1989) we can strongly reject the El-Gamal and Grether model in favor of the structural logit model (P-value 2.5×10^{-4}). Similarly, we can strongly reject the 3 parameter structural probit (see footnote 7) and the 3 parameter reduced form logit model as well (P-values 1.1×10^{-6} and 2×10^{-5} respectively). The noisy Bayesian model is a restricted 1 parameter version of the structural logit model where we allow σ to be freely estimated and restrict β to impose Bayesian beliefs, i.e. $(\beta_0, \beta_1, \beta_2) = (0, 1, 1)$. A likelihood ratio test strongly rejects the hypothesis that subjects are "noisy Bayesians" (P-value 9.2×10^{-12}). The structural logit model is a restricted version of the 5 parameter neural network model and a likelihood ratio test fails to reject the implicit upper-level restriction on the bias term discussed above. (P-value .115). Overall, the structural logit model achieves the lowest value of the AIC (the same as the 5 parameter neural network model) and it is another compelling reason why we use it as the preferred model for our subsequent analysis.

Table 2 presents the maximum likelihood coefficient estimates for the structural logit and probit models. Since the coefficient on $\text{LLR}(d, p_A, p_B, D)$, β_1, is significantly greater

Table 2: Maximum likelihood estimates of the structural logit/probit model parameters

Structural logit model

Parameter	σ	η	β_0	β_1	β_2
Estimate	.38	0	.05	2.38	1.86
Standard error	(.02)	(0)	(.05)	(.28)	(.19)

Structural probit model

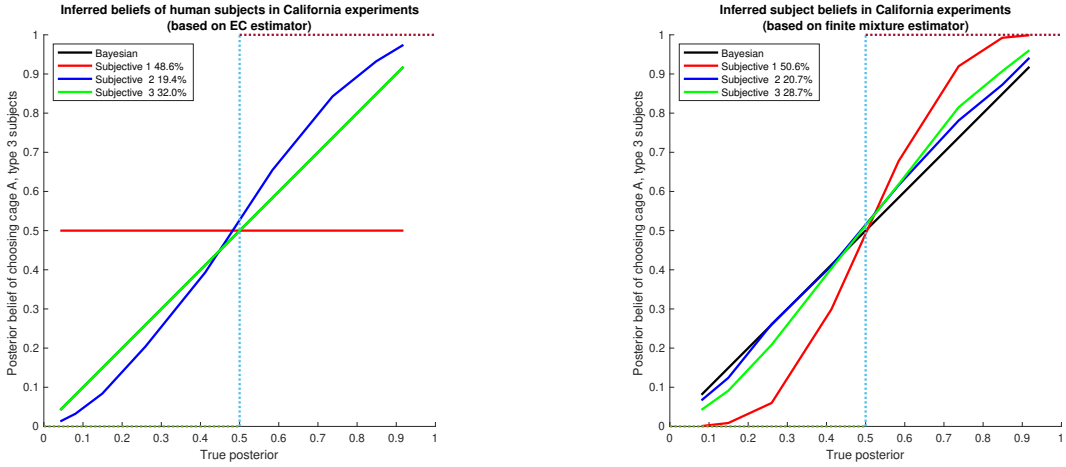
Parameter	σ	η	β_0	β_1	β_2
Estimate	0	1	.03	1.05	.97
Standard error	(0)	(0)	(.02)	(.04)	(.04)

than the coefficient on LPR(π), β_2, the estimation results suggest the typical subject in El-Gamal and Grether's California experiments display the representativeness heuristic, contrary to their finding using their cutoff rule model that the single best model to predict subject behavior is the "noisy Bayesian" model. On the other hand, structural probit model results support the hypothesis that the single best model is the noisy Bayesian model. Neither a Wald test nor a Likelihood ratio test rejects the hypothesis that $\beta^* = (0, 1, 1)$, the values that imply subjective beliefs $\Pi_s(A|d, \pi, p_A, p_B, D, \nu)$ in equation (11) (P-value of likelihood ratio test, .13). However the results in table 1 show that the data reject the structural probit specification in favor of the structural logit model. We can see why this is visually in the right hand panel of figure 3 where the blue line for the 3 parameter reduced-form logit (which fits better than the 3 parameter structural probit but results in a similar CCP) predicts that subjects make no classification errors when the evidence (as captured by the Bayesian posterior) is close to 0 or 1. In actuality, subjects do make significant classification errors at these extremes, and this is why $\sigma > 0$ in the structural logit model and it is able to fit the data significantly better.

Now we show how our conclusions change when we allow for unobserved heterogeneity in subjects' beliefs and behavior. We estimated multiple type models using both the EC algorithm and the finite mixture of types (hereafter abbreviated as FM) method described in section 3.2. We found that AIC is smaller for a specification with $K = 3$ unobserved types compared to $K = 2$ types or the single type specification presented in table 2. Rather than presenting the coefficient estimates, we illustrate the predictions and key findings in a series of graphs below.

Figure 4 compares the types identified by EC and FM in terms of the revealed posterior beliefs. The methods result in different inferences about the proportions of the three types

Figure 4: Inferred Posterior beliefs of California subjects from EC and FM algorithms



as well as their beliefs. The left hand panel shows results from EC algorithm. Type 1 subjects (48.6% of the sample) have a virtually flat posterior. This is due to very small estimates of β and σ, echoing the example we discussed in section 3.1 and the beliefs are similar to those plotted in the example of weak identification of beliefs in figure 2. Type 3 subjects (32% of the sample) have beliefs that are virtually identical to the Bayesian posterior (with the two lines appearing as a single solid green line in the figure) and type 2 subjects (19% of the sample) have beliefs consistent with conservatism since the coefficient β_1 on LLR is significantly lower than β_2, the coefficient of LPR. Their beliefs, plotted by the blue line in the left panel of figure 4, display overconfidence, i.e. they believe cage A is less likely than the true (Bayesian) posterior when the true posterior is less than 1/2, and more likely otherwise.

Though the posterior beliefs of the type 3 subjects are closest to Bayesian posterior beliefs, the degree of idiosyncratic decision noise for these subjects is far higher than for type 1 subjects whose subjective posterior seems nearly flat but is actually strictly increasing as a function of the true posterior. For this reason, we will see below that in terms of decision efficiency, the type 1 subjects are the "most Bayesian" of the three types. That is, both type 1 and 3 subjects are "noisy Bayesians" but type 3 subjects have substantially more noise in their responses with an estimated σ of $\hat{\sigma} = .3$ for type 3 and $\hat{\sigma} = 1.9 \times 10^{-6}$ for the type 1 subjects.

The right panel of figure 4 plots the inferred beliefs from the FM approach. The results appear rather different than those from the EC algorithm: the Type 1 subjects

(50.6% of the sample) conform to the representativeness heuristic, since the estimated coefficient on LLR is approximately twice as large as the coefficient of LPR. The other two types identified by the FM method can be classified as "noisy Bayesians" since the coefficients on LLR and LPR are not significantly different from each other. However type 2 subjects are substantially noisier than type 3 subjects: the estimated σ for the former is .75 compared to only .16 for the latter. Thus, we will designate the type 3 subjects as the "most Bayesian" type discovered by the finite mixture method. Overall the optimized likelihood from the FM approach, -1655 is considerably below the value yielded by EC algorithm, -1543, reflecting the advantage of EC as a "pure strategy" for identifying types compared to FM which is akin to using a "mixed strategy". Despite these differences, we show that both methods provide similar overall results and conclusions.

Figure 5: CCPs implied by EC and FM models of California subjects

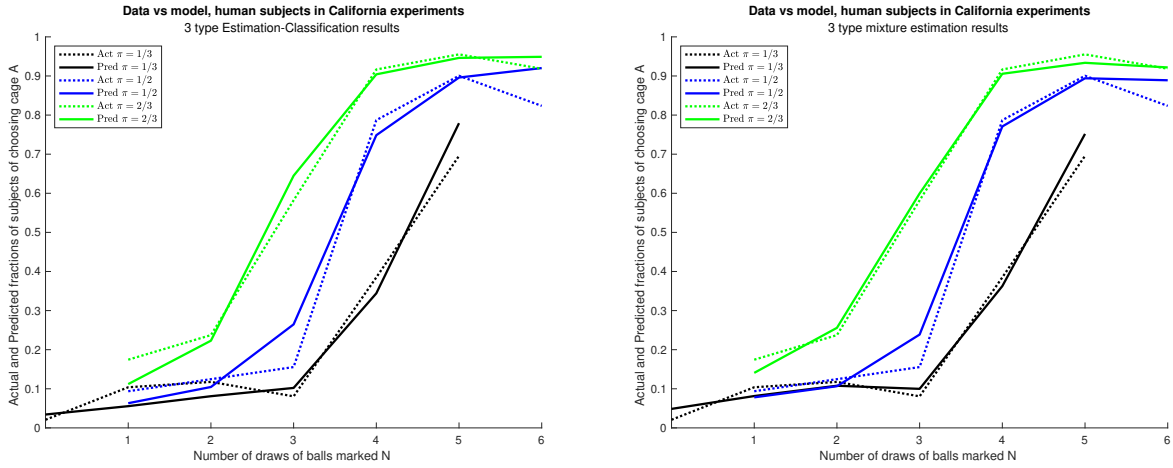
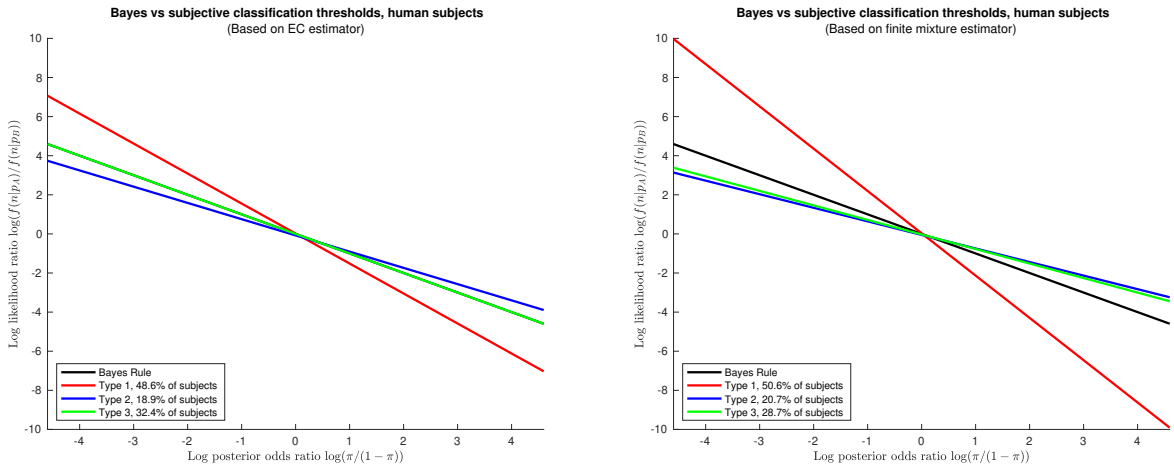


Figure 5 compares the overall CCPs (for all subjects, not broken down by type), plotted for each of the three priors used and for all observed outcomes for d. We see that EC and FM give very similar overall fits to the data.

Figure 6 plots the classification thresholds for each type of subject. These are hyperplanes of the form $\hat{\beta}_{0,k} + \hat{\beta}_{1,k}\text{LLR} + \hat{\beta}_{2,k}\text{LPR} = 0$ for each type $k \in \{1, 2, 3\}$ where cages A and B are subjectively equally likely. We also plot the classification threshold for Bayes Rule, the black line with a slope of -1. Even though figure 4 creates an impression that EC and FM approaches lead to different conclusions about the proportions of subjects with different subjective posterior beliefs, figure 6 shows that the implied classification thresholds are quite similar. In particular the subjects whose beliefs are consistent with

Figure 6: Classification thresholds implied by EC and FM models of California subjects



Representativeness are the ones with the steeper classification threshold in each graph and these are the Type 1 subjects that are estimated to be about 50% of the subject pool by both methods. The main difference is that the type 3 subjects identified by the EC method have a classification threshold that is virtually identical to Bayes Rule, whereas there is a clearer difference for the type 3 subjects identified by the FM method.

Figure 7: Accuracy and efficiency of California subjects implied by EC and FM algorithms

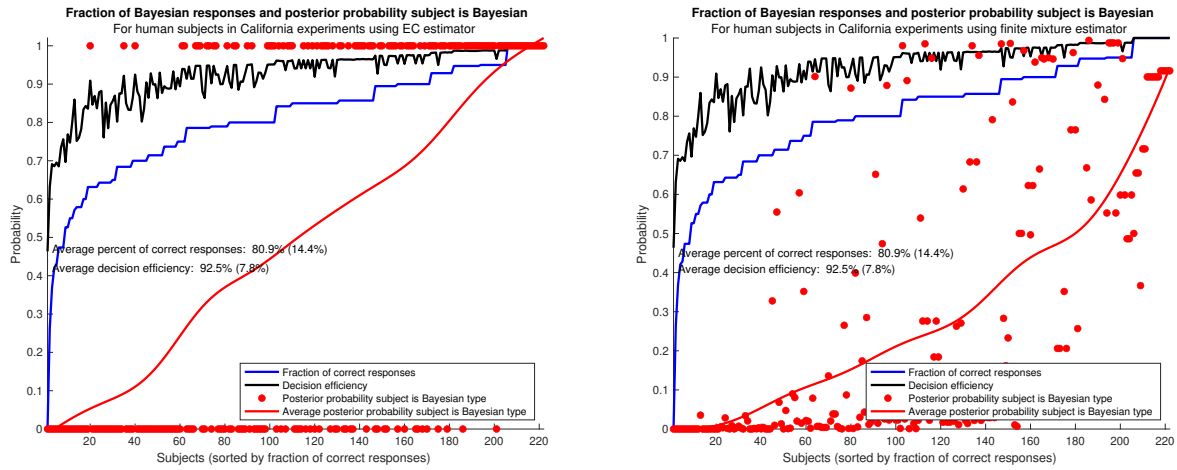


Figure 7 plots subject-specific accuracy and efficiency scores as well as the posterior probability that the subject is the "Bayesian type" implied by each subject's choices and the estimated structural logit model, as estimated using the EC and finite mixture methods, respectively. The accuracy score is simply the fraction of each subject's choices that coincide with the choices of perfect Bayesian decision maker. The efficiency score is the sum of expected wins in the T_s trials each subject s participated in to the corresponding

wins for a perfect Bayesian, i.e. the ratio ω_s given by

$$\omega_s = \frac{\sum_{t=1}^{T_s} \left[\Pi(A|d_{ts},\pi_{ts})^{y_{ts}} + [1-\Pi(A|d_{ts},\pi_{ts})]^{(1-y_{ts})}\right]}{\sum_{t=1}^{T_s} \left[\Pi(A|d_{ts},\pi_{ts})^{y^*_{ts}} + [1-\Pi(A|d_{ts},\pi_{ts})]^{(1-y^*_{ts})}\right]}. \tag{20}$$

where d_{ts} and π_{ts} are the trial outcomes and priors, respectively, and y_{ts} is an indicator for subject s's choice of cage A in trial t, and y^*_{ts} is the choice a perfect Bayesian would make in the same trial.

The red dots in each figure are the posterior probabilities implied by the structural logit model that each of the 222 subjects is the "most Bayesian type" (i.e. type 1 for the EC algorithm and type 3 for the finite mixture model). Since the EC algorithm classifies each subject to be the most likely type, these posterior probabilities are either 0 or 1 depending on whether the most likely type of subject s is type 1. In the case of the finite mixture method, we can use the estimated probabilities of each type as a "prior probability" of that type and use the subject-specific likelihood to compute a posterior probability for each type $\tau \in \{1,2,3\}$, denotes by $\Pi(\tau|y_s,d_s,\pi_s)$ and given by

$$\Pi(\tau|y_s,d_s,\pi_s,\hat{\theta}) = \frac{\hat{P}(\tau)L(y_s,d_s,\pi_s|\tau,\hat{\theta})}{\sum_{k=1}^{3}\hat{P}(k)L(y_s,d_s,\pi_s|k,\hat{\theta})}. \tag{21}$$

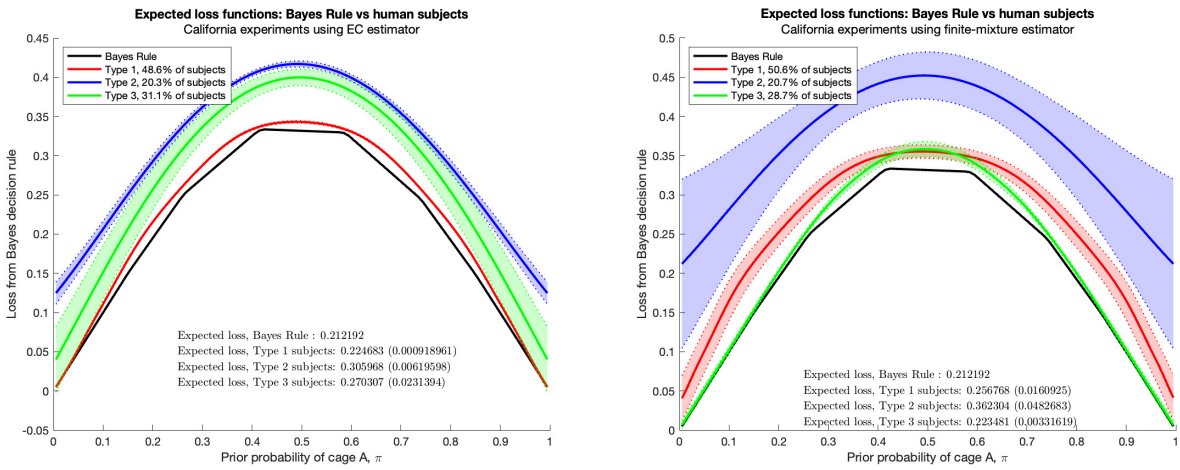
where y_s is the sequence of choices by subject s in the T_s trials, and d_s and π_s are the corresponding outcomes and priors for these trials, and $\hat{P}(k)$ is the estimated fraction of type k subjects, and $L(y_s,d_s,\pi_s|k,\hat{\theta})$ is the subject-specific likelihood for subject s at the estimated parameter values $\hat{\theta}$ assuming the subject is type k.

The right hand panel of figure 7 plots the posterior probabilities, for each subject, that they are the "most Bayesian type" (type 3 per the discussion above). Of course these posterior probabilities are not all zero or one for the finite mixture estimator. To enable better comparison the red line in both panels plots the local average probability that the subject is the most Bayesian type and we can see that this probability is monotonically increasing in the fraction correct responses (accuracy) and is also strongly positively correlated with subject-specific decision efficiency, though variation across subjects in efficiency is not as great as the variation in accuracy. This is a reflection of the observation we made in the introduction that a subject with lower accuracy need not have significantly lower efficiency if the trials where their choices deviate from Bayes Rule are mostly the

"hard cases" where the Bayesian posterior is close to 1/2.

Figure 8 plots the loss functions for the three types of subjects implied by the EC and FM models as a function of the prior π. We see that again the implied loss functions are similar, though the estimated standard error bands are larger for the loss functions estimated by FM, especially for the type 2 subjects (blue line, the noisier subset of noisy Bayesians). The EC method predicts that the type 1 subjects have the lowest loss function, whereas the FM method predicts that the type 3 subjects have the lowest loss.

Figure 8: Loss functions implied by EC and FM models of California experiments



We calculated expected win probabilities using the empirical distribution of π for all three types. For these experiments a Bayesian would, on average, predict the correct cage with probability 70%. According to the EC algorithm type 1 subjects predict the correct cage with an average probability of 68%, so the estimated average decision efficiency score is $\omega_p = 97.5\%$ (standard error, 0.7%). The FM algorithm predicts that the less noisy Bayesians (Type 3 subjects) are the most efficient decision makers, with a decision efficiency score of 94.7% (1.5%). Both the EC and FM algorithms predict that the least efficient decision makers are the type 2 subjects with decision efficiencies of 87% and 82%, respectively. Overall, for all three types of California subjects, both the EC and FM algorithms predict a decision efficiency score of 93% with estimated standard errors of 0.7% and 1.2%, respectively. Wald tests strongly reject the hypothesis that subjects are fully efficient decision makers either individually for each type or overall as a group.

4.2 Reanalysis of Wisconsin Experiments

Now we reanalyze data from the "Wisconsin experiments" conducted by El-Gamal and Grether (1999). They recruited 79 student subjects from the University of Wisconsin-Madison and employed a two stage experimental design to test for *context effects* by altering the "California design" used in El-Gamal and Grether (1995) (where $D = 6$ and $p_A = 2/3$ and $p_B = 1/2$) to a new "Wisconsin design" ($D = 7$, $p_A = .4$ and $p_B = .6$). The experiments were conducted on two successive days. On the first day approximately half the subjects began with the 6 ball California design and then switched to the 7 ball Wisconsin design on the second day, whereas for the other half of subjects this order was reversed to see if the ordering of the designs affects subjects' choices.

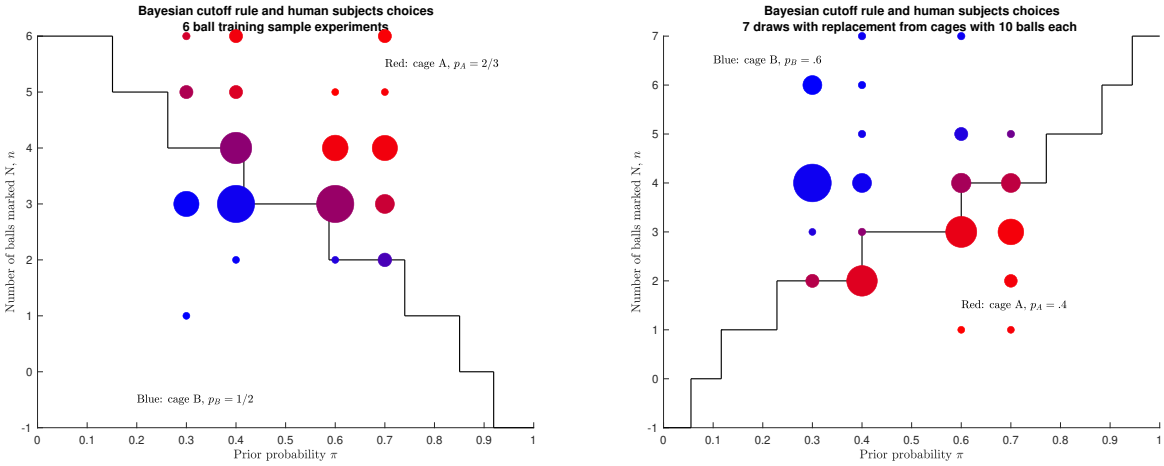
The motivation for this design is twofold: 1) by drawing an odd number of balls, they made it harder for subjects to rely on the "representativeness heuristic" in their choice of cage A or B, and 2) by reversing the proportions of balls marked N they reversed the classification rules. That is, in the 7 ball Wisconsin design larger values of d constitute stronger evidence for cage B, not cage A as was the case in the California design. These design changes could potentially change subject behavior, whereas they have no effect on the optimal Bayesian decision rule which is a function of $\text{LLR}(d, p_A, p_B, D)$ and $\text{LPR}(\pi)$. In fact, the structural logit model predicts that the design change is fully accounted for in the likelihood ratio term entering Bayes Rule. It follows that even if the subject is not Bayesian (e.g. the β coefficients do not equal the values $\beta = (0, 1, 1)$), as long as the structural coefficients are unaffected by the change in experimental design, then so is the implied behavior of these subjects.

From our perspective, the Wisconsin experiments provide an opportunity to treat the switch from the 6 ball California design to the 7 ball Wisconsin design as a *policy change* to study how well the structural logit model is able to predict subjects' choices under a different experimental design (data generating mechanism). To do this we estimate a multi-type version of the structural logit model for all 79 subjects using the 6 ball California design trials as the *training sample* and use the 7 ball Wisconsin design trials as the *evaluation sample.* This approach also allows us to conduct a likelihood ratio test for *structural stability,* i.e. to test whether the coefficients of the structural logit model are invariant across the two experimental designs. As we noted in the introduction, El-

Gamal and Grether (1999) concluded (using their structural probit model) that subjects did appear to use different decision rules in the 6 ball and 7 ball designs. Evidently not being able to rely as easily on the representativeness heuristic lead a greater fraction of subjects to use decision rules that more closely approximate a noisy version of Bayes rule in the 7 ball trials.

Figure 9 summarizes the data on subject choices in the 6 and 7 ball trials. The size of each circle is proportional to the number of observations and the color of the circles represents the share of subjects choosing cage A or B. Thus, deep blue circles are those where nearly all of the subjects chose cage B, deep red circles are those where nearly all the subjects chose cage A, and the purple colored circles are those where there was a mix of subjects choosing cage A and B, where we mixed red and blue in proportion to the fraction of subjects choosing A and B.

Figure 9: Data and classification thresholds in the Wisconsin experiments



We plot the cutoffs implied by Bayes Rule as a function of the prior probability π to make it easier to assess whether subjects' responses are consistent with the optimal decision rule: at points on one side of the cutoff line cage A is optimal and on the other side cage B is the optimal choice. We see how the change in experimental design changed the threshold from downward sloping in π in the 6 ball design to upward sloping in the 7 ball design. It is evident that choices are on average consistent with Bayes Rule. Subjects become "confused" (indicated by purple circles) for trial outcomes near the classification threshold or "indifference curve." The circles further away from it are colored deep red or blue, so these are the easy cases where subject choices largely accord with Bayes Rule.

It seems there is less confusion in the 7 ball design but this may be an artifact of fewer observations laying on near the classification threshold compared to the 6 ball design.

Table 3 presents the maximum likelihood estimates from the FM algorithm for the 4 parameter specification of the structural logit model where $\eta = 0$.[12] We stopped the FM algorithm when it reached $K = 3$ types even though the likelihood ratio test indicated that $K = 4$ types is a significantly better fit because with only 79 subjects, there would be too few subjects assigned to some of the types in the 4 type specification, raising the possibility of model overfitting and less reliable inferences.

Table 3: FM estimates of the structural logit model for 6 ball experiments, $LL = -654.25$

Parameter (std error)	Type 1 ($\hat{\lambda}_1 = .32$)	Type 2 ($\hat{\lambda}_2 = .24$)	Type 3 ($\hat{\lambda}_3 = .44$)
σ (noise parameter)	.07 (.08)	.22 (.06)	.32 (.05)
β_0 (bias/intercept)	.14 (.18)	.06 (.11)	-.11 (.08)
β_1 (LLR(n) coefficient)	1.48 (1.91)	1.25 (.36)	2.48 (.67)
β_2 (LPR(π) coefficient)	1.40 (1.81)	2.27 (.70)	1.39 (.36)
P-value for $H_o : \beta_0 = 0, \beta_1 = \beta_2$	.74	.02	.01
P-value for $H_o : \beta_0 = 0, \beta_1 = \beta_2 = 1$	.015	.003	.015

Of the three types, type 1 is the most "Bayesian" and also the least "noisy". The estimated noise parameter σ is less than a third of the value estimated for the other two types. There is no significant overall bias in the beliefs of any of the three types as evidenced by the fact that all three estimates of the bias term β_0 in subjective posterior beliefs in equation (11) is insignificantly different from zero. The next to the last row in the table presents the P values of a Wald hypothesis test that $\beta_0 = 0$ and $\beta_1 = \beta_2$, and we see that there is no evidence against this for type 1 subjects but the hypothesis is rejected for type 2 and type 3 subjects. The type 2 subjects are those whose beliefs are consistent with the Representativeness heuristic, i.e. they put less weight on the prior information via LPR relative to the sample information via LLR. The type 3 "conservative subjects" do the opposite.

The last row of the table shows that we can strongly reject the hypothesis that any of the three types are "noisy Bayesians" i.e. who satisfy the hypothesis $H_o : \beta_0 = 0, \beta_1 =$

[12] Estimation results from the EC algorithm result in similar overall conclusions.

$1, \beta_2 = 1$, though with possible noise affecting their choices, $\sigma > 0$. However because the type 1 subjects satisfy the restriction that $\beta_0 = 0$ and $\beta_1 = \beta_2$, their implied decision rule is nearly optimal, as we show below. So at least in terms of their decision rule or *choice behavior* the type 1 subjects are close to optimal even if their subjective posterior beliefs may be a distorted version of true Bayesian beliefs.

Figure 10: Subjective vs Bayesian Posterior Beliefs, 6 ball experiments

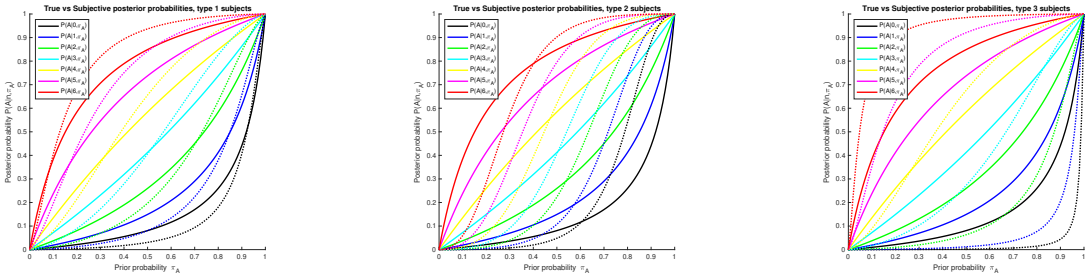


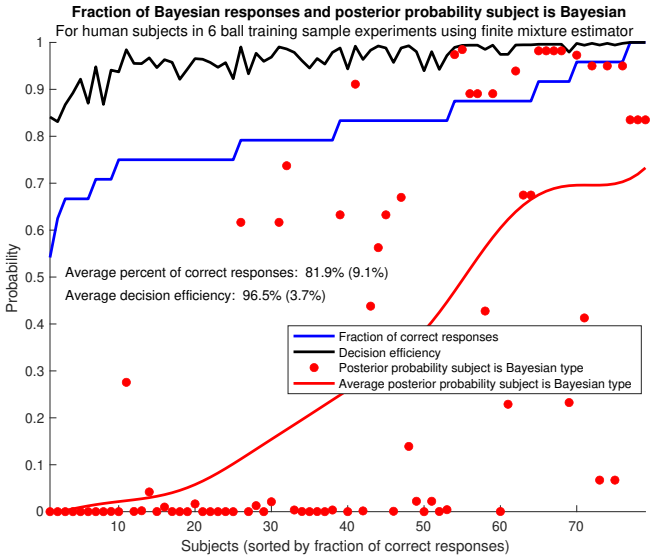
Figure 10 plots the estimated subjective posterior beliefs for the three types of subjects (dotted lines) and compares them to the true Bayesian beliefs (solid lines) for all possible combinations (π, d) where $\pi \in [0, 1]$ and $d \in \{0, 1, \ldots, 6\}$. It is visually apparent that the subjective posterior beliefs of the Type 1 subjects are closest to Bayesian, whereas there are obvious distortions in the estimated posterior beliefs of the Type 2 and 3 subjects. Type 2 subjects display a pattern of *overconfidence* overestimating the probability of A relative to Bayes Rule for sufficiently high values of π and under-estimating it for sufficiently low values of π. Type 3 subjects display a different type of overconfidence. For high values of d ($d \in \{0, 1, 2, 3\}$) these subjects' subjective beliefs systematically under-estimate the probability of A relative to Bayes Rule for *all* values of π, but for high values of d ($d \in \{5, 6\}$) this reverses and they systematically over-estimate the probability of A relative to Bayes Rule, for all $\pi \in [0, 1]$.

The distortions in subjective posterior beliefs imply suboptimal choices even in the absence of "noise" in their ultimate decisions. If σ were zero, the optimal decision rule can be described by a cutoff rule $\overline{\pi}(d)$ that differs for each realized value of d. These cutoffs occur at the intersections of the solid or dashed curves and a horizontal line at a probability value of 1/2. It is optimal for the subject who observes information (π, d) to choose cage B if $\pi < \overline{\pi}(d)$ and cage A if $\pi \geq \overline{\pi}(d)$. From this it is evident that the Type 1 subjects have the least distortion in the implied optimal (noise-free) decision rules, whereas the Type 3 subjects have the most distorted decision rule. We show below that

this implies a significantly higher loss for type 3 subjects relative to type 1 subjects.

Before showing this, we present figure 11. The blue curve in this figure plots the fraction of "correct" responses for each of the 79 subjects, i.e. the fraction of their responses that coincide with the optimal choice of a Bayesian decision maker given the same information (π_t, d_t) in each trial t. We see that the fraction of correct responses ranges from a low of 54% to a high of 100% with an average of over 81% correct responses. The black and red curves help relate this simpler "percent correct" metric of performance to Bayesian behavior. The black curve plots subject-specific decision efficiencies, i.e. the ratio of the sum of each subject's expected win probabilities in all trials to the corresponding total for a perfect Bayesian decision maker. We see that average efficiency is quite high, 96.5%, which exceeds the average of 81.9% correct responses of these subjects. This is due to the fact that most of the "mistakes" that these subjects made were on the hard cases where the true posterior probability was close to 1/2, and thus, mistakes for these cases are not very costly for these subjects.

Figure 11: Fraction of Correct Responses

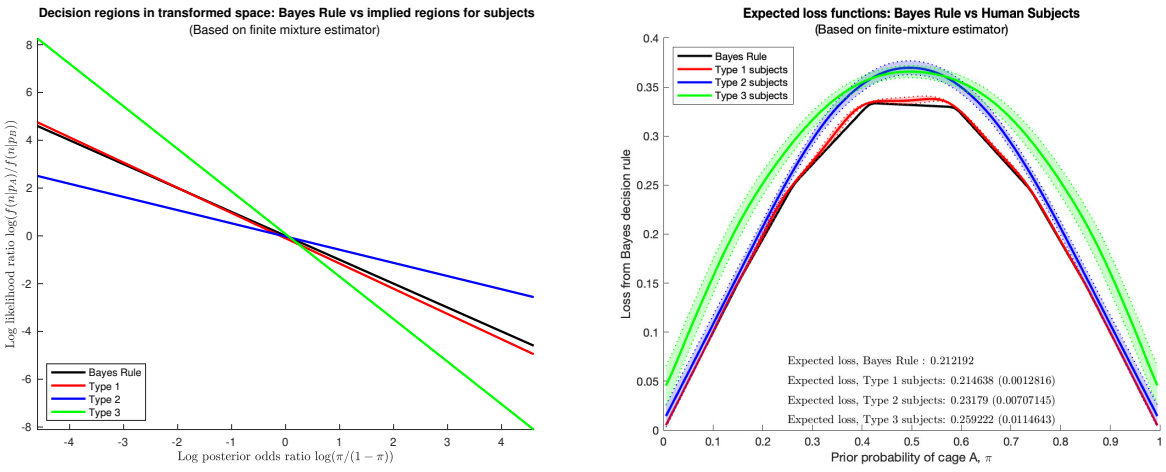


The red curve in figure 11 plots the posterior probability (implied by the FM estimates) that each subject is a type 1 "noisy Bayesian" and the solid red curve is a local linear regression showing the local average fraction of noisy Bayesians ordered by percent correctly answered. We see a strong positive relation between "correctness" and the probability of being a Type 1 subject, ranging from a low of 0% for subjects who only answered 54% of the trials correctly to a high of 70% for subjects who answered 100%

correctly.[13]

Figure 12 plots the classification thresholds (hyperplanes) for each of the three types of subjects along with the prior-specific expected loss functions $L_P(\pi)$, see equation (3). Recall that along these lines the subject believes cages A and B are equally likely, so they each demarcate the decision regions each type of subject would make their classification decisions in the absence of noise (i.e. when $\sigma = 0$). Here it is obvious that the decision hyperplane of the Type 1 subjects essentially coincides with the Bayesian classification threshold, but the hyperplane for the Type 2 subjects is significantly less steep and the hyperplane for the Type 3 subjects is significantly more steep than the classification hyperplane of a Bayesian. This allows us to readily assess the regions where subjects will make classification errors relative to what an optimal Bayesian decision maker would choose. For example in the triangular region to the left of $\text{LPR}(\pi) = 0$ below the red and black lines and above the blue line, for values of (LLR, LPR) in this region a type 2 subject chooses cage A but a Bayesian (or type 1 subject) chooses cage B. Similarly the triangular region below the green line and above the red and black lines to the left of $\text{LPR}(\pi) = 0$, represents misclassifications by Type 3 subjects: they choose cage B whereas a Bayesian (or type 1 subject) chooses cage A.

Figure 12: Classification Thresholds and Expected Loss Functions, 6 ball experiments



[13]Of course, it is possible to get 100% due to unobservable random factors affecting subjects' responses and this is why the posterior probability of subjects who answered 100% correctly is not 1. In fact, the posterior probability of being a Type 1 subject can be rather low even if the subject answered 100% correctly depending on the particular sequence of trial values $\{(\pi_t, d_t)\}$ the subject is asked to classify. If most of these trials correspond to the hard cases true posterior values close to 1/2, i.e. $\Pi(A|\pi_t, d_t) \simeq 1/2$, then it is harder to distinguish whether the subject is a noisy Bayesian or some other type. This is reflected in a higher posterior probability that the subject could have been a type 2 or 3 subject.

The right hand panel of figure (12) plots the expected conditional loss functions $L_P(\pi)$ for the three types of subjects. The black curve plots the optimal Bayesian loss function $L_{\delta^*}(\pi)$ that necessarily minorizes the loss functions for the human subjects by Lemma L1. We see that the red expected loss curve for the type 1 subjects nearly coincides with the black Bayesian loss function, indicating that the Type 1 subjects are nearly optimal Bayesian decision makers, even if their subjective posterior beliefs are somewhat distorted relative to Bayes Rule. However the green and blue curves are significantly higher than the black and red expected loss curves, showing that the Type 2 and 3 subjects are using distinctly suboptimal decision rules.

The most suboptimal decision makers are the type 3 subjects whose $L_P(\pi)$ loss function majorizes the other three expected loss functions. Note that the other loss functions are close to zero at values of π equal to 0 or 1, whereas the green curve is significantly positive at those values similar to what we found for the poorest performing types of California subjects. This is a reflection of the high amount of "decision noise" σ for the type 3 subjects that we noted in our discussion of the estimation results in table 3. For example the model predicts that a type 3 subject will choose cage A about 5% of the time even when $\pi = 0$ when there is no chance that cage A was used to draw the sample, and conversely for values of π near 1.

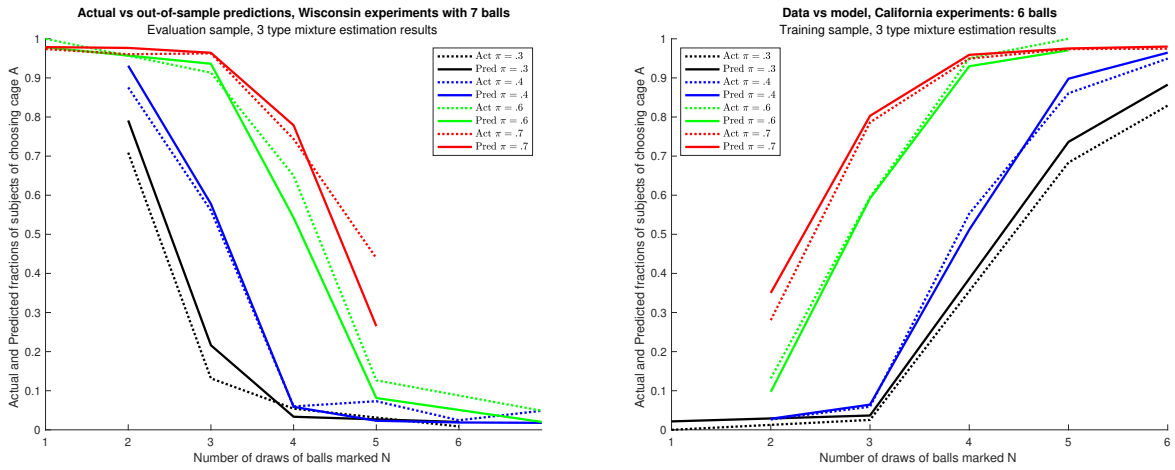
We also calculated expected win probabilities using the empirical distribution of π for all three types. The EC algorithm predicts that the type 1 (less noisy Bayesians) have a decision efficiency score of 99.5% (standard error 0.2%), whereas type 2 (Conservative) subjects have a decision efficiency of 96.2% (standard error 0.9%), and type 3 (Representativeness) subjects have an efficiency index of 94.5% (0.1%). While we can strongly reject the hypothesis that any of the three types is a fully optimal decision maker, the overall efficiency of the Wisconsin subjects is quite high, 96.5% (0.5%) which is higher than the 93% overall efficiency score for the California subjects.[14]

Finally, we consider how well the structural model can predict a "policy change" i.e. a change in the experimental design from the 6 ball California design to the 7 ball Wisconsin design. We illustrate this in figure 13. The left panel of the figure shows the in-sample fit, i.e. it compares the predicted CCPs from the structural model to the non-parametric estimates of the CCPs for subjects in the 6 ball "training sample". Note that we incorpo-

[14]Overall efficiency of the Wisconsin subjects in the 7 ball design was 95.5% (0.6%).

rated the subject-specific unobserved heterogeneity in the model-predicted CCPs in this figure: that is, we computed the structural logit CCPs by averaging the subject-specific CCPs where each subject was assigned their most likely type from the output of the EC estimation algorithm. We plot separate lines for the 4 separate priors used in these experiments, $\pi \in \{.3, .4, .6, .7\}$ where the dashed lines plot the non-parametric CCPs as a function of n and the solid lines are the predictions of the structural logit model.

Figure 13: Predicted vs Actual CCPs in Training and Evaluation Samples



The right hand panel illustrates the ability of the structural logit model to predict how subject behavior changes in the $D = 7$ ball evaluation sample. The 4 curves are now *downward sloping in* d and of course this is due to a change in binomial parameters from $(p_A, p_B) = (2/3, 1/2)$ in the California design to $(p_A, p_B) = (.4, .6)$ under the Wisconsin design. We see that the structural logit model provides fairly accurate predictions of the dramatic shift in subjects' decision rules under the two experimental designs, suggesting that the structural logit model provides a good approximation to subject behavior.

However there is a more subtle aspect in which subjects' decision rules might have changed that constitutes evidence against the "structural stability" of the structural logit model. That is, if the structural logit model was a completely correct model of subject behavior, then if we were to estimate the parameters of the structural logit model using the 7 ball evaluation sample, the structural parameters θ should not be statistically significantly different from the parameters estimated in the 6 ball training sample. However we find that when we do use the 7 ball experiments as the estimation or training sample, there is a significant change in the structural parameter estimates: a smaller fraction,

23%, of subjects are classified as "noisy Bayesians". The noisy Bayesians also exhibit significantly greater noise in their responses compared to the type 1 subjects identified using the 6 ball training sample, with $\hat{\sigma} = .43$. Further, we find that type 1 and 3 subjects have relatively low inefficiency but type 2 subjects exhibit high inefficiency due to the large values of σ noted above. In the 7 ball data, we find that the inefficient, high noise subjects constitute 23% of the sample, whereas the other 77% of subjects have higher efficiency due to lower estimated values of σ.

Overall the fraction of subjects using "efficient" decision rules increased from 32% in the 6 ball training sample to 77% when we use the 7 ball experiments as the training sample. A likelihood ratio test strongly rejects the hypothesis that subjects were using the same decision rules in both the 6 ball and 7 ball experimental designs: P-value .002.[15] However the average inefficiency of all Wisconsin student subjects under the 7 ball design is 10.7% (standard error, 1.4%), which is significantly larger than the 8.1% average inefficiency of all subjects for the 6 ball design. Apparently the switch to the 7 ball design reduced the ability of subjects to rely on the representativeness heuristic. This may have induced more subjects to expend more mental effort resulting in more efficient decision rules. Despite this, the 23% of the sample who continued to use the representativeness heuristic also had significant noise affecting their responses, so the significantly higher inefficiency among this group outweighed the improvement in efficiency among the other subjects, causing overall inefficiency to increase in the 7 ball Wisconsin design.

4.3 Reanalysis of Holt and Smith Experiments

The identification problem for subjective beliefs using only binary choice data discussed in section 3.1 suggests the need for caution in drawing conclusions about the fraction of subjects who have subjective posterior beliefs that are well approximated by Bayes Rule, though we can be confident our inferences on overall inefficiency of human subjects since this measure is based on the CCP which is non-parametrically identified.

In this section we reanalyze experiments reported in Holt and Smith (2009) that

[15] When we estimated the 3 type structural model via the FM method using the 221 California subjects we obtained roughly similar results as we obtained for the Wisconsin subjects when we estimated their parameters for the same 6 ball California design. We found that 21% of the California subjects were "noisy Bayesians", 50% overweight LLR and hence show behavior consistent with the representativeness heuristics, and 29% overweight LPR and hence exhibit behavior consistent with conservatism. However the estimated σ parameter for the noisy Bayesians was $\sigma = .74$ and this high level of noise made these subjects the most suboptimally behaving subgroup among the California subjects.

directly elicited subjective posterior beliefs. This was done via the *Becker-DeGroot-Marshak* (BDM) mechanism which incentivizes rational subjects to truthfully report their subjective posterior probabilities. They conducted two separate experiments: one at Holt's laboratory at the University of Virginia involving 22 subjects, and a second one done via the Internet involving 30 subjects. In both experiments the design parameters involved fixing $p_A = 2/3$ and $p_B = 1/3$ but varying the number of draws D and the priors across multiple trials for each subject. D took values from $\{0, 1, 2, 3, 4\}$ and π varied from $\{1/3, 1/2, 2/3\}$. Here we focus on the reanalysis of the first experiment with 22 subjects and, given space constraints, summarize the key findings from our analysis of their second web-based experiment in a footnote.

The BDM mechanism was implemented as follows: after seeing the prior π and the result of the random drawing d from the selected cage/cup, subjects were asked to report a probability $p_r \in [0, 1]$ that determines a payoff they would receive in a second stage gamble. This gamble, denoted by $\tilde{G}_R$, involves drawing a random probability $\tilde{p} \sim U(0, 1)$ and paying the subject a monetary reward of R according to the following rule: if $\tilde{p} < p_r$ the subject receives R if the observed sample was drawn from cup A, otherwise if $\tilde{p} \geq p_r$ the subject receives R with probability $\tilde{p}$. It is not hard to show that the subject's expected payoff from reporting p_r in this second stage BDM lottery is

$$E\{\tilde{G}_R | p_r, d, \pi, p_A, p_B, D\} = R\left[\frac{1 - p_r^2}{2} + p_r \Pi_s(A|d, \pi, p_A, p_B, D)\right], \tag{22}$$

where $\Pi_s(A|d, \pi, p_A, p_B, D)$ is the subjective posterior probability that the sample was drawn from cup A given the information (d, π, p_A, p_B, D). Further, it is easy to see that the report p_r that maximizes expected payoff is $p_r^* = \Pi(A|d, \pi, p_A, p_B, D)$.

Note that the BDM mechanism incentivizes *truthful reporting* but not necessarily *correctness* of subject reports. The latter depends on their capability as "intuitive Bayesians" and their ability to do internal mental calculations (i.e. the extent to which they do "fuzzy math"). A drawback of the BDM mechanism is that it can be confusing to subjects and potentially harder for them to determine the optimal report p_r than to determine the posterior probability of cup A. Holt (2019) notes that "The use of incentivized elicitation procedures is the norm in research experiments, but there are some problems." One of the problems is that BDM relies heavily on the presumption of rationality of the human

subjects, including being able to calculate the expected payoff in equation (22) as a function of p_r and then realize that the payoff maximizing report is $p_r^* = \Pi(A|d, \pi, p_A, p_B, D)$. If subjects are not so good in probability and math, the second stage BDM mechanism might actually mislead or confuse them and would then add extra "decision noise" into the experimental results.

Indeed, Holt (2019) acknowledges that the "BDM procedures may be difficult for subjects to comprehend." (p. 110). The instructions to subjects in the Holt and Smith (2009) experiments instructed them that their payoff is maximized by truthfully reporting their subjective posterior. To the extent subjects trusted and followed this advice the BDM mechanism may not necessarily confuse subjects or add extra "decision noise".

Subjects in Holt and Smith's experiments were not asked to make an additional binary choice of which cup they believed the observed sample was more likely to have been drawn from. It seems quite reasonable to assume that if subjects would have been asked to make such a choice (perhaps incentivized by an additional payment for selecting the correct cup) that they would have chosen cup A if $\Pi(A|d, \pi, p_A, p_B, D) > 1/2$ and cup B otherwise.[16] This implies that $\sigma = 0$ in our structural logit model specification in equation (12), and we use this to generate the binary decision rule implied by subjects' beliefs in our analysis of implied inefficiency of subjects' decision rules below. Any inefficiency in subjects' decision making is then due to "calculational noise" ν and/or bias and incorrect weights β on the constant, LPR and LLR in the formula for subjective posterior beliefs $\Pi_s(A|d, \pi, p_A, p_B, D, \beta, \nu)$ in equation (11).

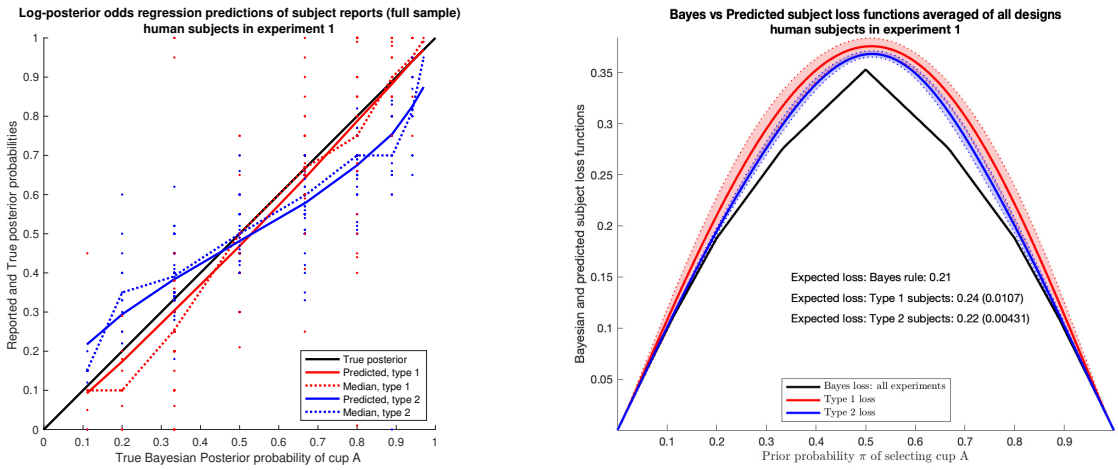
We estimated β and the parameter η under the assumption that $\nu \sim N(0, \eta^2)$ by maximum likelihood using the log reported prior odds ratio regression specification in equation (10).[17] We also estimated multi-type versions of these models using the EC algorithm and finite mixture approaches. We find a strong improvement in the likelihood from going from 1 to 2 types, but we stopped at $K = 2$ types because of the relatively small number of subjects (22 and 24 in experiments 1 and 2, respectively).

[16] Or randomly guess if $\Pi(A|d, \pi, p_A, p_B, D) = 1/2$.

[17] A small fraction of subjects reported posterior probabilities of 0 or 1 for which the log reported posterior odds ratio is undefined. Rather than exclude these observations we estimated a truncated regression specification where we assume that a value of 0 is reported when the subjective posterior is lower than some lower threshold $\underline{p}$ and report a value of 1 when it exceeds an upper threshold $\overline{p}$. It is not hard to show that the maximum likelihood estimates of these additional parameters are the min and max of the subset of reported subjective posterior values that are strictly in the $(0, 1)$ interval. We verified that all conclusions are robust to simply excluding the observations with reports of 0 or 1, or recoding them to arbitrary values such as .00001 and .99999.

The results from the EC and finite mixture models are quite similar, both in the parameters and the estimated fractions of each type. We omit the actual parameter values and describe the results more informally and graphically below. We will discuss the results from the finite mixture method, though the results from the EC algorithm are virtually identical. For experiment 1 the two types can be described as 1)"noisy Bayesians" (45% of the subjects) and 2) conservatives, i.e. those who put more weight on LPR than LLR. For the noisy Bayesian subjects, we cannot reject the hypothesis that $(\beta_0^*, \beta_1^*, \beta_2^*) = (0, 1, 1)$ (i.e. values that result in Bayesian beliefs), though there is significant "calculational noise" as evidenced by the large and significant estimate of $\hat{\eta} = 0.91$ (std error (0.14)). We strongly reject this hypothesis for the type 2 "conservative" subjects and find a small bias against choosing cup A. However we note that the type 2 subjects are far less noisy than the type 1's with an estimated value of $\hat{\eta} = 0.40$, less than half the value we estimated for type 1 subjects.

Figure 14: Predicted vs Actual Median Beliefs and Loss Functions: Holt Smith Experiment 1

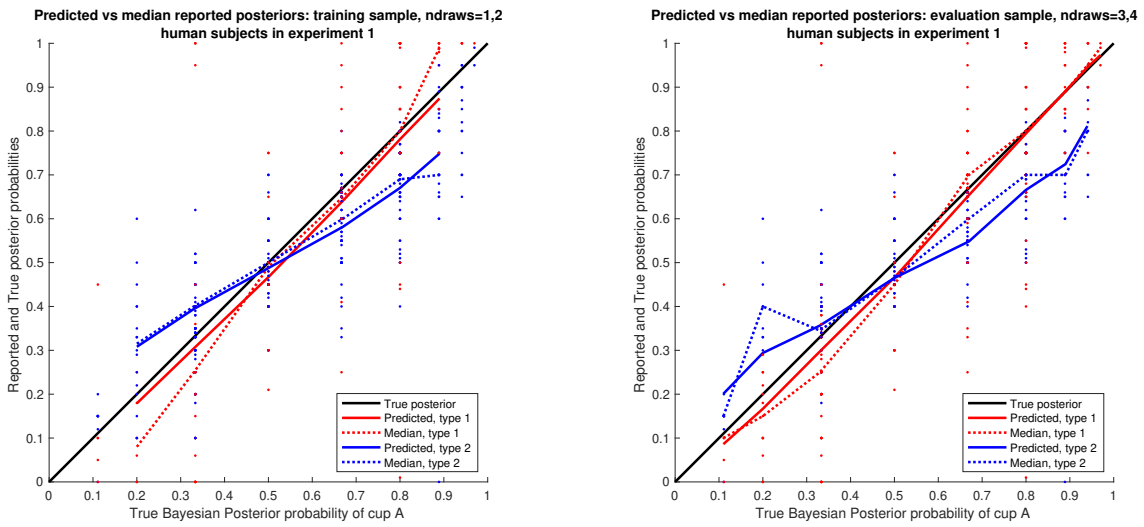


The left hand panel of figure 14 provides a scatterplot of the actual subject responses where the x axis corresponds to the implied true Bayesian posterior probability for the (LPR, LLR) pairs for each subject and trial. For reference the black 45 degree line is the true Bayesian posterior probability and the dotted lines are the median values of the subjects' responses. Using the EC algorithm we can classify each subject as type 1 (noisy Bayesian) or 2 (conservative), and the dotted lines plot the median of the subject responses for these two types. Finally the solid red and blue lines are the predicted medians of subject responses from the estimated structural logit model. We see that the

model fits the data well and that the Bayesian subjects have median posterior beliefs that are quite close to the 45 degree line. However the median beliefs of the type 2 conservative subjects increase less steeply than the true Bayesian posterior does, reflecting "underconfidence" i.e. when the true posterior exceeds 1/2 the median beliefs of type 2 subjects is below the Bayesian posterior, and conversely when the true posterior is less than 1/2. As we noted in the discussion in section 3.1 underconfidence itself does not necessarily imply inefficiency of the decision rule. Instead, inefficiency is more sensitive to the calculational noise, i.e. the value of η.

The right hand panel of figure 14 plots the loss functions implied by decision rules that pick cup A when $\Pi_s(A|d, \pi, p_A, p_B, D, \nu) > 1/2$ and cup B otherwise. Due to the higher value of η for the type 1 (noisy Bayesian) subjects, their expected win probability is lower than the win probability of the type 2 conservative subjects (blue line). When we compute the average win probability for all subjects weighted by the empirical distribution of (π, D) values used in the design of experiment 1, we find an average decision efficiency score of 96% with a standard error of 0.7%. A Wald test strongly rejects the hypothesis that the human subjects are fully efficient decision makers. However overall, the decision efficiency for these subjects is high and in line with results we find for subjects in El-Gamal and Grether's California and Wisconsin experiments.

Figure 15: Stability of beliefs: 1,2 vs 3,4 trials from Holt Smith Experiment 1



Since the design of experiment 1 has the total number of draws from the cups D varying from trial to trial over the values $D \in \{0, 1, 2, 3, 4\}$, this provides an opportunity

to evaluate the ability of the structural logit model to predict "out of sample beliefs" and test for structural stability. We estimated the model for values of $D \in \{1, 2\}$ only as the "training sample" and then compare how well the estimated model predicts median reported beliefs for values of $D \in \{3, 4\}$ that we refer to as the "evaluation sample." The left hand panel of figure 15 plots the actual median and fitted median beliefs for type 1 and 2 subjects in the training sample and the right hand panel show both for the evaluation sample, where the model predictions are based on parameters estimated in the training sample where $D \in \{1, 2\}$ only. Visually, the structural model does a good job in predicting median beliefs of the subjects in the evaluation sample.

However we do observe that the fractions of subjects estimated to be type 1 or type 2 subjects changes when we restrict the sample. For example in the training sample, we find only 27% of the subjects are type 1 noisy Bayesians, whereas in the evaluation sample we estimate that 55% of subjects are noisy Bayesians. Further the amount of noise, as estimated by η is higher in the training sample ($\hat{\eta} = 1.22$) compared to the evaluation sample ($\hat{\eta} = .74$). Thus, we find evidence of "structural instability" and a likelihood ratio test strongly rejects the hypothesis that subjects are using the same decision rule in both the training and evaluation samples. Evidently, the training sample where $D \in \{1, 2\}$ is a harder decision problem than the evaluation sample where $D \in \{3, 4\}$ and subjects change their decision rules to cope with this. The parameter for the standard deviation of the calculational errors is higher for the harder decision problem. Of course, this is also true for a Bayesian: the loss function for a Bayesian when $D\{1, 2\}$ majorizes the loss function for a Bayesian when $D \in \{3, 4\}$.[18]

4.4 Conclusions from the re-analysis of human subject data

Below we list the key takeaways from our reanalysis of human subject data from El-Gamal and Grether (1995), El-Gamal and Grether (1999) and Holt and Smith (2009).

[18] We also analyzed data from 30 subjects in Holt and Smith's experiment 2 which was conducted online. The overall conclusions are similar to those from our reanalysis of experiment 1, except that the EC algorithm no longer finds any noisy Bayesians: 62% of subjects put excessive weight on the prior and the remaining 38% put too much weight on the data. The level of calculational noise for these subjects, η, is also significantly higher. The higher degree of noise in subjects' reports implies significantly higher loss and thus lower efficiency. Average decision efficiency for all subjects in all trials in experiment 2 was 91% (0.8%), lower than the 93% efficiency of the subjects in El-Gamal and Grether's 6 ball California experiments and lower than the 96% efficiency of human subjects in Holt and Smith's experiment 1.

1. We confirm their strong rejection of the hypothesis that *all subjects* behave as Bayesian decision makers (or even noisy Bayesians). Using the EC algorithm and finite mixture methods we find substantial unobserved heterogeneity among different subjects as well as significant idiosyncratic random noise that affects their responses.
2. However subsets of subjects in the El-Gamal and Grether experiments are classified as noisy Bayesians, ranging from 32% in the Wisconsin 6 ball experiments to 77% in the Wisconsin 7 ball experiments. In the California and Wisconsin experiments subjects provide only binary responses so a "revealed belief approach" is required to make these inferences and we showed that the identification of beliefs is fragile. In the experiments run by Holt and Smith posterior beliefs were elicited via the BDM mechanism, enabling us to directly identify their posterior beliefs. In experiment 1 we find that 45% of subjects are classified as noisy Bayesians, but in their web-based experiment 2, none of the subjects are.
3. Even though subject beliefs are difficult to identify in the El-Gamal and Grether experiments, their decision rules are identified. We find high overall decision efficiency, ranging from 93% of the optimal Bayesian win probability in their California experiments, to 96.5% for subjects in their Wisconsin experiments that used the 6 ball California design.
4. Holt and Smith did not ask subjects to make a binary choice of the more likely cup, but we can impute this choice based on their elicited posterior probability of cup A. We find significant calculational noise in the reported posterior probabilities, but the overall efficiency of a decision rule that selects cup A if the subjective posterior probability exceeds 1/2 is still high: 96% for the subjects in their experiment 1 and 91% in their web-based experiment 2.
5. The structural logit model provides a good approximation to the different decision rules used by subjects and "out-of-sample" forecasts of how their behavior changed in responses to changes in experimental design. However our analysis confirms the main conclusion of El-Gamal and Grether (1999) that subjects appear to use different decision rules in different experimental designs. Thus, the structural logit model can only approximate the more complicated underlying cognitive processes governing human subject responses.

5 Optimality and revealed beliefs of AI subjects

We will now analyze the performance of AI subjects collected from a series of experiments that replicated human studies, using various versions of ChatGPT: beginning with the earliest GPT-3.5, moving on to GPT-4, and culminating in the latest version, GPT-4o. These iterations demonstrate significant advancements in general-purpose AI capabilities.

We begin by introducing the design of the prompts used to conduct experiments with GPTs. We first replicate the Wisconsin experiments, where subjects are required to

perform binary classification tasks, as outlined in El-Gamal and Grether (1999). We also elicit the subjective posterior beliefs of AI subjects using prompts very similar to the experiments in Holt and Smith (2009). Although it is less clear whether there are specific incentives for AI subjects to exert less effort or misreport, we retain the BDM mechanism in our prompts to ensure that our design aligns as closely as possible with the original human experiments.

We use the structural logit model developed in Section 3 to interpret the GPT data and test whether subjects are Bayesian. We do this to provide a parallel analysis of human and GPT subjects, and due to the flexibility and superiority of the structural logit model relative to other behavioral models as we discussed in Section 4. We also allow for unobserved heterogeneity using both the EC and finite mixture methods. In our GPT experiments, we intentionally injected variability in their temperatures to proxy for unobserved heterogeneity in human subjects. The temperature parameter in a GPT controls the scale of random "noise" in their responses and thus it is akin to heterogeneity in the extreme value σ parameter in equation 13.[19] We show that the EC and finite mixture methods can effectively detect and control for the variability in temperature, which in the case of GPT subjects is actually an observed covariate rather than unobserved heterogeneity.

We use the accuracy and decision efficiency metrics defined in section 2 to compare the performance of GPT and human subjects. We find that GPTs are subject to biases that lead to suboptimal decisions. However, we observe a rapid growth in overall ability across successive generations of GPT subjects, causing their performance to improve from sub-human levels in GPT-3.5 to human-like proficiency in GPT-4, and ultimately to superhuman and nearly perfect Bayesian classifications in the latest version, GPT-4o.

5.1 Prompt Design for Experiments in ChatGPT

Similar to Chen et al. (2023), we conduct our experiments by submitting inquiries through the public OpenAI application programming interface (API). Using APIs allows us to conduct massive experiments in a timely and cost-effective manner. We formulate the

[19] In GPTs, the temperature refers to a parameter that controls the randomness of the model's responses. Lower temperature results in more deterministic and focused responses, making the model more likely to choose the most probable next words. A higher temperature increases randomness, allowing for more varied responses. The default temperature for the three versions of GPTs we consider here is 0.7.

prompt by drawing an analogy between GPTs and human experiments. Each version of a GPT represents a distinct school where human experiments have been conducted. We consider the varying temperature settings for GPTs as analogous to different students or human subjects. Once we define the schools and subjects, we repeated submitting inquiries using our prompts through APIs, as each human subject participates in the trials in different experiments.

Appendix B outlines the algorithm we developed to implement the experiments, which involves looping over different versions of GPTs (schools), temperature settings (subjects), experiments, and trials. We then parse the responses from ChatGPT and collect its choices of either Cage A or B in the Wisconsin experiment,[20] or the reported subjective probability in the case of the Holt and Smith experiment. Appendix C provides further details about the prompts we use.

One notable difference between the prompt in our study and that of Chen et al. (2023) is that we allow GPTs to report their reasoning process, rather than solely providing the conclusion as done in Chen et al. (2023).[21] We deviate from them for two reasons. First, in the daily use of GPTs, people rarely impose such restrictions. Second, when answering our questions, GPTs typically answer them step by step. This chain of thought may potentially enhance GPT performance, as suggested by Wei et al. (2022).

5.2 Analysis of the Wisconsin Experiments in GPTs

We begin by analyzing the binary classification results from the Wisconsin experiments conducted with the GPTs. Following our re-analysis of human subjects, instead of pooling the data from the 6 and 7 ball design we focus on the 6 ball design and use the results from the 7 ball design as a validation set to test "structural stability" of the structural logit model.

Figure 16 summarizes the choice probabilities in the first day experiment. Each circle represents a trial, with prior probabilities, π, on the $x-$axis and number of N-draws, n,

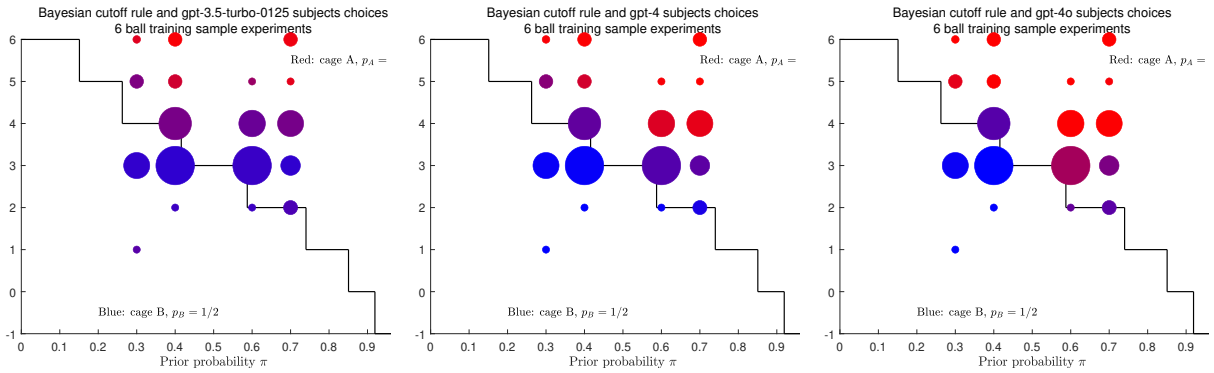
[20] Occasionally, GPTs may stop prematurely before providing an answer regarding the choice between A and B. In such cases, we resubmit the same inquiry until the GPT delivers a classification. We consider this process to be natural, as it mirrors our everyday use of ChatGPT—if it fails to provide a satisfactory answer due to an unexpected stop, we simply ask again. In our implementation, it takes a maximum of 5 iterations to resolve any missing answers in our experiment.

[21] For example, the prompt for the investment experiment in Chen et al. (2023) ends with "First please *only* tell me the number of points for investing Asset A, then please *only* tell me the number of points for investing Asset B." Including the word *only* forces GPTs to report only the final conclusion without the reasoning process.

on the $y-$axis. The size of each circle indicates the number of choices made by subjects, while the color reflects their selections: deep blue indicates more subjects chose cage B, and deep red indicates more chose cage A. The black step function represents the optimal integer cutoffs calculated using Bayes' rule. Given a prior probability, it is optimal for a subject to choose cage A if n exceeds the cutoff, and to choose cage B if it falls below.

For Bayesian decision-makers, we expect red circles above and blue circles below the Bayesian cutoffs. GPT-3.5 subjects make more errors than human subjects and advanced GPTs, as illustrated by large purple circles above the cutoff in Figure 16. For example, in trials with 4 N-draws and prior probabilities 0.6 and 0.7, all should choose A (pure red), but many GPT-3.5 subjects choose B, mixing red with blue and creating the purple circles. In trials with 3 N-draws and a prior probability 0.6, almost all GPT 3.5 subjects choose Cage B. In contrast, human subjects mix between Cages A and B, aligning more closely with the Bayesian posterior probability of 1/2.[22]

Figure 16: Choices of GPT Subjects: 6-ball Experiments



(a) GPT 3.5 (b) GPT-4 (c) GPT-4o

The accuracy of GPT-4 and GPT-4o is much improved. The two large purple circles disappear in Figure 16 and 16. However, neither GPT-4 nor GPT-4o are perfect Bayesian decision makers. Mistakes still occur in certain trials, indicated by two purple circles for GPT-4 and one for GPT-4o, all positioned above but close to the cutoff. [23]

We then estimate a multiple-type structural logit model using the FM method.[24] The

[22]The distribution of mistakes among trials are also different between GPT and human subjects. While a small fraction of human subjects make errors distributed across nearly all trials, GPT-3.5 tends to be uniformly accurate in some trials, with its errors concentrated in others.

[23]We find a similar pattern in the second-day experiment that the accuracy improves with more advanced GPTs. We also note that, same as human subjects, GPTs make less mistakes in the 7-ball experiment than in the 6-ball experiment.

[24]We also estimate the model using the EC algorithm. Although the EC algorithm typically identifies more types than the FM method, we prefer the FM method for two reasons. First, some types identified by the EC

estimates detect unobserved heterogeneity among GPT subjects, with different subjective posterior beliefs (determined by the belief parameters β's) and level of idiosyncratic noise (captured by the extreme value scale parameter σ) associated with each type.

The FM method identifies two types for GPT-3.5 subjects. Figure F1(a) plots the estimated subjective posterior beliefs against true posteriors for both types.[25] Both types deviate from the 45-degree line, showing non-Bayesian beliefs. However, when subjective posteriors are on the same side of 1/2 as Bayesian posteriors, they can result in identical decisions to those of Bayesian decision makers. 58% of subjects are type 1, aligning perfectly with Bayesian decisions when true posteriors are below 1/2, but often differing when subjective posteriors bounce to zero as true posteriors cross 1/2. Type 2 subjects have smaller wiggles in their subjective posteriors, but can still jump to different directions than the Bayesian posterior, leading to different decisions than Bayesian decision makers.

Turning to the estimated σ parameters, we find that type 1 subjects have smaller values than type 2 subjects. The difference in the estimated noise parameters reflects the known heterogeneity in GPT 3.5 subjects injected by our random assignment of temperatures. The mean temperature for type 1 subjects is 0.43, nearly half that of the noisier type 2 subjects. This supports the notion that a higher "temperature" corresponds to greater "noise" in subject responses and demonstrates the effectiveness of our algorithm in detecting unobserved heterogeneity.

The FM method only identifies a single type of GPT-4 and GPT 4o subjects, respectively. The estimated posterior beliefs in Figure F1(b) and F1(c) are almost flat, but they do have a tiny positive slope, leading to the same CCPs as Bayesian decision makers.[26] The seemingly flat beliefs conceal the differences in subjective beliefs between GPT-4 and GPT-4o. Figure 17 shows the implied classification hyperplanes, whose slopes equal the negative ratio of coefficients for LLR ($\widehat{\beta}1$) and LPR ($\widehat{\beta}2$). The Bayesian classification

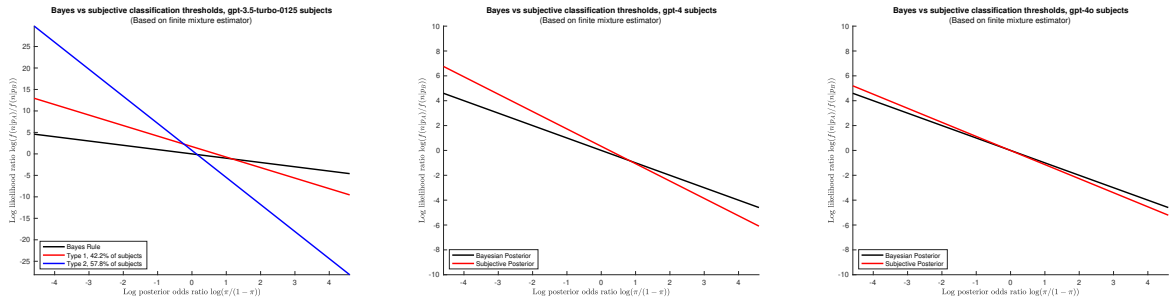
algorithm may be redundant due to subtle parameter differences or a small number of subjects in certain types. Second, starting from the estimates of EC algorithm, the FM method can always consolidate these into fewer types.

[25] Subjective posterior beliefs, $\Pi_s(A|d, \pi, p_A, p_B, D, \nu)$ in equation 13, depend on the trials, represented by (d, π). Rather than plotting against the two-dimensional trial specifications, we summarize (d, π) by calculating the corresponding Bayesian posterior probabilities, offering a one-dimensional summary. Recall that we assume $\nu = 0$ in the Wisconsin experiment.

[26] We estimate the model from multiple starting values, all converging to the same single-type estimate. To further validate that GPT-4o acts as a single-type "noisy Bayesian" decision maker, with subjective posterior beliefs showing a slight positive slope against true posteriors, we estimate a noisy Bayesian model with β parameters constrained to $[0, -1, -1]$, estimating only σ. The likelihood ratio test does not reject the noisy Bayesian model.

hyperplanes, which have a slope of -1, are in black. Although the β values are small, the ratios are close to -1, so GPT-4o subjects assign nearly equal weights to LLR and LPR and their subjective classification hyperplane is closer to the Bayesian hyperplane than GPT-4, which places more weight on LPR than on LLR.

Figure 17: Estimated classification hyperplanes for GPT Subjects: 6-ball Experiments



(a) GPT 3.5 (b) GPT-4 (c) GPT-4o

We next turn to evaluate the model fit and "structural stability" of the multiple-type structural logit model when applying to data generated by GPTs. Figure F2 compares the model-predicted CCPs for GPT-3.5 with the data from the estimation sample (on the left) and the second-day experiment as a validation sample (on the right). Although the in-sample fit is good,[27] the out-of-sample predictions, unlike those for human data, are quite poor. This is because the choice probabilities in GPT 3.5 lack monotonicity with respect to both LPRs and LLRs, which can be a very weak requirement for rationality. The structural logit model, however, imposes such monotonicity and therefore, is unable to capture the complicated nature of mistakes taken by GPT 3.5. We can extend the simple structural logit model to capture such non-monotonicity by incorporating higher order terms of LLRs and LPRs. However, due to limited observations, more flexible specifications may lead to overfitting. Therefore, we choose to retain our simple structural logit models.

Likelihood ratio tests reject the hypothesis of structural stability of the model with GPT-4 and GPT-4o subjects. However, the rejection is weaker for more advanced GPTs. This suggests that while the structural logit model doesn't perfectly capture GPT behavior, its accuracy improves as subjects become more rational.[28]

[27] There are two exceptions. It underestimates the probability of choosing Cage A with $N = 2$ at a low prior probability of $\pi = 0.3$, and it significantly overestimates the probability with $N = 5$ at a high prior probability of $\pi = 0.6$.

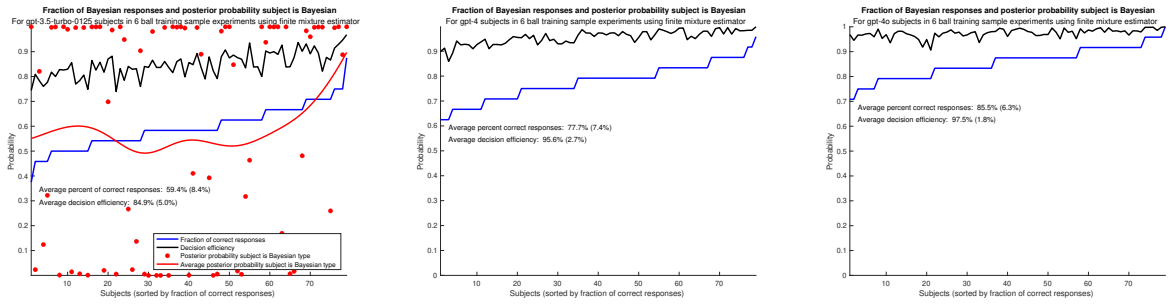
[28] We conjecture that for GPT-o1, as it converges to nearly perfect Bayesian behavior, we will see structural

Next we compare the performance of the three generations of GPTs by calculating accuracy and decision efficiency as in Section 2. The blue step function in Figure 18 shows that accuracy varies from below 40% to 90% among GPT 3.5 subjects, with a very low average of just 59.4%. The accuracy curve moves up for GPT-4 and rises further for GPT-4o, highlighting the improvement in accuracy across successive generations of GPTs.

The rapid increase in rationality is further supported by the measure of decision efficiency, plotted in Figure 18 by black curves, which shift upwards from 18 to 18. Decision efficiency generally increases as subjects achieve higher accuracy, but there are noticeable fluctuations at a local level. They arise because a subject with a slightly higher correct response rate may struggle on "easy" cases—where the true posteriors are more extreme and easier to differentiate—while randomly guessing correctly in "hard" cases, where the true posterior is around 1/2. As a result, a subject with higher accuracy may show lower decision efficiency. GPT-4 and GPT-4o have much smaller wiggles than GPT-3.5 reflecting the lower level of estimated noise, σ, affecting their responses.

When multiple types are identified, as is the case with GPT 3.5, Figure 18 selects the most Bayesian type and calculates the posterior probability that a subject belongs to this type, represented by the red dots. The red Bayesian-type curve plots the average posterior probability of the most Bayesian type, estimated using a local linear regression of these red dots. The fluctuation in the red dots and the non-monotonicity of the Bayesian-type curve show that higher accuracy in a subject doesn't necessarily make it more likely to be a Bayesian type.

Figure 18: Percent correctly answered for GPT Subjects: 6-ball Experiments



(a) GPT 3.5 (b) GPT-4 (c) GPT-4o

stability and the structural logit model (the subcase of a near noiseless Bayesian) will almost perfectly describe its behavior.

Table 4 summarizes the overall performance of humans and various GPTs in the Wisconsin 6 ball design experiments, averaging across all estimated types. In terms of both efficiency and accuracy the GPT-3.5 subjects perform significantly worse than humans. Although humans are comparable to GPT-4, they are less efficient than GPT-4o. This underscores the rapid transition in GPT performance from "subhuman" to "superhuman" with just a few upgrades over a relatively short period.

The bottom row in Table 4 reports the number of subject types detected by the FM method. We find that there are fewer types identified for the later versions of GPT, suggesting that these subjects are converging a single type that is very close to a Bayesian decision maker. We find more heterogeneity among human subjects than for GPTs, whose behavior is more uniform, despite our intentional introduction of heterogeneity via temperature.

Table 4: Performance of GPTs and humans in the 6-Ball Experiments

	GPT 3.5	GPT-4	GPT-4o	Humans
Efficiency	84.9	96.0	97.5	96.5
	(0.6%)	(0.3%)	(0.2%)	(0.5%)
Accuracy	59.4	77.7	85.5	81.9
	(8.4%)	(7.4%)	(6.3%)	(9.1%)
No. of Types	2	1	1	3

5.3 Analysis of the Holt and Smith Experiments in GPTs

Now we turn to an analysis of elicited posteriors using the Holt and Smith design with GPT subjects. These data allow us to directly observe subjective posterior beliefs which are otherwise challenging to identify when subjects report only binary choices.

We controlled for subject heterogeneity in the GPT subjects using the FM method.[29] As with the analysis of human data, we restrict it to at most two types due to the limited number of subjects involved in the experiment.

We replicate both the experiments run at the University of Virginia (experiment 1) as well as online (experiment 2). In contrast to our reanalysis of human data, we find that the estimated subjective posterior beliefs are very similar for GPT subjects in the two experiments. This is not surprising, as humans may be more easily distracted in online than in offline experiments, as suggested by the larger variance in the error terms. Such

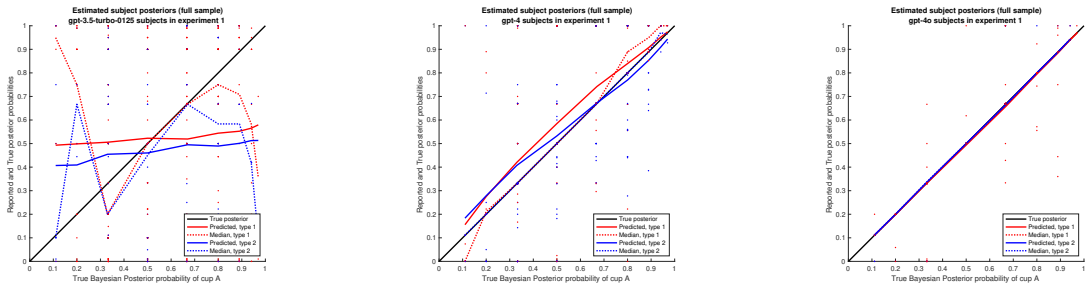
[29] We obtain very similar results using the EC method.

differences do not appear in experiments with GPTs. Therefore, we focus on presenting the results from the experiment 1 where the average human decision efficiency was higher, 96% compared to 91% for human subjects in experiment 2.

Figure 19 plots, for the 3 versions of GPT subjects, their reported posterior probabilities against the corresponding Bayesian posterior probabilities, each represented as a dot.[30] The EC algorithm identifies two types of subjects for all three GPTs, denoted by red dots (type 1) and blue dots (type 2), respectively.

The dotted lines illustrate the median reported posterior probabilities derived from the data. It is evident that each successive generation of GPT subjects report posterior beliefs that increasingly align with the true Bayesian posterior, as evidenced by their proximity to the 45-degree Bayesian line. GPT-3.5 frequently makes numerical mistakes in calculating its posterior, with both the frequency and severity of these inaccuracies escalating when true posterior probabilities are more extreme, particularly in the ranges below 0.2 and above 0.8, where discrepancies are especially pronounced. For GPT-4, the median subjective posterior probabilities of type 1 subjects closely align with the Bayesian 45-degree line, except that they may slightly overestimate the posterior probabilities when the true posteriors are close to 1. Type 2 subjects also approximate Bayesian decision-making in most trials, except for those with true posterior probabilities near 0, where subjects tend to underestimate probabilities, and near 1, where they tend to overestimate. The median subjective posteriors for both types of subjects in GPT-4o are almost identical to the Bayesian decision makers.

Figure 19: True vs Estimated Subjective Posterior Probabilities



(a) GPT 3.5 (b) GPT-4 (c) GPT-4o

The solid red and blue lines in Figure 19 represent the model-predicted median of

[30] In one trial, a subject from GPT 3.5 reported a posterior probability of 2, which is excluded from Figure 19. However, we do not exclude this observation from our sample.

subject responses, constructed from the estimated β's. Consistent with the findings from the Wisconsin experiments, we observe a markedly improved model fit as subjects approach Bayesian decision-making. GPT-3.5 exhibits numerous errors in its responses, which makes it challenging for the simple structural logit model to capture the full range of atypical behaviors.

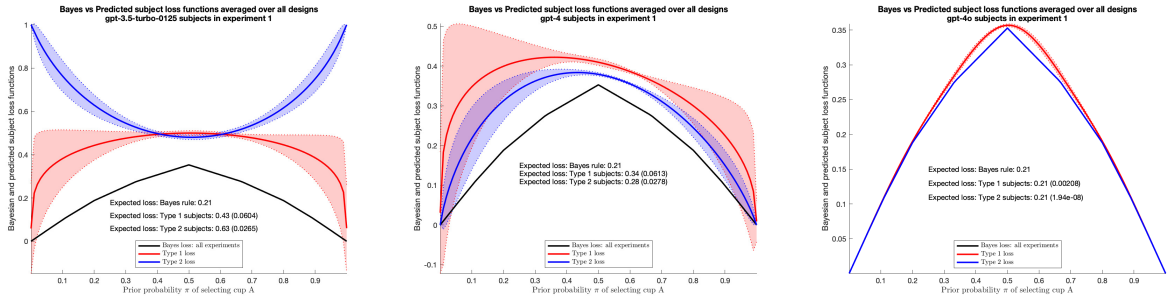
The structural logit model fits much better in GPT-4 and GPT-4o. However, both subject types identified in GPT-4 consistently overestimate posterior probabilities when the true posteriors are below 0.7, while underestimating them otherwise. Both types assign less weight to LPR, however, the type 1 subjects are closer to Bayesian, as evidenced by a smaller difference between the coefficients on LPR and LLR.

The model-predicted and data posterior probabilities both coincide with the Bayesian 45-degree line, illustrating an almost perfect model fit. Both types of GPT-4o subjects put almost equal weights on LPR and LLR and have negligible bias, so their posterior beliefs are close to Bayesian. The two types are different mainly in their estimated noise parameters.

Beyond the parameter β's in the subjective beliefs, the estimated noise parameter $\widehat{\eta}$ reveals another aspect of improvement, namely, the level of calculation noise in their subjective beliefs reduces in each successive generation. For GPT 3.5, the degree of noise is large for both types, with $\widehat{\eta} = 2.8$ for type 1 subjects and $\widehat{\eta} = 2.1$ for type 2 subjects. In contrast, the level of noise is significantly smaller for GPT-4, with $\widehat{\eta} = 0.8$ for type 1 subjects and $\widehat{\eta} = 1.6$ for type 2 subjects. By the time we get to GPT-4o there is almost no estimated noise for type 2 subjects, $\widehat{\eta} = 0.0025$, which means that effectively they are perfect, noiseless Bayesians. In contrast, type 1 subjects have a slightly larger noise of $\widehat{\eta} = 0.4$, making them noisy Bayesian decision makers.

To facilitate a comparison of decision efficiency across different GPTs and between GPTs and human subjects, we calculate the expected loss functions for various prior probabilities of selecting cage A and plot them in Figure 20. Both the proximity of subjective beliefs to Bayesian beliefs and smaller noise levels contribute to a smaller expected loss. Because the subjective beliefs are closer to Bayesian beliefs and there is less noise affecting the beliefs of the more advanced GPT versions, the expected loss functions are closer to the efficiency benchmark as shown in moving from Figure 20 to Figure 20.

Figure 20: Loss Functions by Types



(a) GPT 3.5 (b) GPT-4 (c) GPT-4o

Finally, table 5 summarizes the performance of human and GPT subjects in Holt and Smith's experiment 1. The table reports our decision efficiency and accuracy scores as well as a new measure of "accuracy", namely, the R^2 of a regression of the reported posterior on a constant and the true posterior. Values closer to 1 for each of these measures can be interpreted as "more Bayesian". We see a steady progression in efficiency and accuracy of the GPT subjects from GPT 3.5 to GPT 4o. The human subjects are superior to GPT 3.5 and 4 in terms of efficiency, but GPT-4o shows superhuman performance, especially in terms of decision efficiency and accuracy as measured by R^2, reflecting the fact that reported posteriors by GPT subjects are significantly closer to the true Bayesian posterior on average than the noisier human subjects.

The EC algorithm finds 2 types of subjects for each version of GPT and human subjects, and in each case the key difference is the level of estimated noise (captured by the standard deviation parameter η representing "calculational errors" in the structural logit model). Except for GPT 3.5, both types of subjects are "noisy Bayesians" but one of the types makes significantly larger calculational errors than the other. Among the GPT-4o subjects, 45% are classified as essentially perfect noiseless Bayesians with 100% decision efficiency. For human subjects, 45% are Bayesians also, but due to the greater level of noise in their response their decision efficiency is lower, 95%.

For GPT 4 and GPT 4o this noise correlates directly with the "temperature" parameter we set to generate heterogeneity among the GPT subjects. For example for GPT-4o, the noisier subjects had average temperature of .75 (std error .30), twice the average temperature of the less noisy subjects identified by the EC algorithm. Thus, as we would expect higher temperature implies noisier responses by GPT subjects and this noise leads to lower efficiency and accuracy. We can obtain even better performance for the GPT-4o

subjects by reducing the temperature parameter.

Table 5: Performance of humans and GPT in the Holt-Smith Experiments

	GPT 3.5	GPT-4	GPT-4o	Humans
Efficiency	75.0	93.0	99.4	96.0
	(1.0%)	(0.4%)	(0.3%)	(0.7%)
Accuracy	58.1	84.0	98.2	87.4
	(0.1)	(0.1)	(0.02)	(0.1)
R^2	0.7	41.8	88.0	63.5
No. of Types	2	2	2	2

5.4 Context Effects

Echoing David Grether's criticism that humans are more influenced by context effects in "real world" situations, we test whether GPT subjects exhibit similar biases. We randomly sample 100 trials in the Wisconsin 6-ball experiment for each GPT model, where temperatures were near the median, and replace *bingo cages and balls* in the prompt as *ponds and fishes*, where humans may have more experience and GPTs might learn from that. Table 6 compares the accuracy of the original bingo experiments with the reframing pond experiments. Context effects are rejected by the t-test for all three GPT models we considered, as none are even close to the conventional significance levels.

Table 6: Accuracy Comparison between Bingo and Pond Experiments

Model	Bingo	Pond	t-stat	p-value	N
GPT-3.5	0.63	0.69	-0.84	0.40	100
GPT-4	0.85	0.83	0.45	0.66	100
GPT-4o	0.94	0.92	0.63	0.53	100

5.5 Conclusions from the analysis of GPT subjects

Below we list the key conclusions from our analysis of the El-Gamal and Grether Wisconsin experiments and the Holt and Smith experiments using GPT subjects:

- We find a rapid improvement in the accuracy and decision efficiency in successive versions of GPT, from sub-human performance for GPT 3.5, to approximately human for GPT-4, and super-human performance for GPT-4o.
- The estimates of the multiple-type structural logit model reveal three key

aspects of this improvement in performance over successive generations of GPT: (i) decreasing bias, (ii) increasingly equal weight on the data and the prior, and (iii) lower levels of idiosyncratic calculational and decision noise.

- We find less unobserved heterogeneity for GPTs than humans in the Wisconsin experiments, especially for GPT-4 and 4o, where we find only one type of subject, noisy Bayesians. In the Holt and Smith experiment, 40% of subjects are classified as perfectly Bayesian, while the remaining 60% are best described as noisy Bayesians.
- As in the case of human subjects, a likelihood ratio test of "structural stability" rejects the hypothesis that the structural logit model is the correct model of the choices of GPT subjects. However, the P-values of these rejections increase with each successive version of GPT, indicating that the structural logit model provides an increasingly good approximation to the behavior of successive generations of GPT.

6 Analysis of Errors from AI Subject Response Text

Most econometric models treat humans as black boxes and apart from some research from neuroscience, we do not know exactly how humans process information. However, from the textual responses of GPT, we have the unique advantage of observing the reasoning of GPT subjects, opening the door to analyze where GPTs make mistakes. There are two challenges in such an analysis: the errors made by GPTs can be highly diverse, and the textual responses are not well-structured.

We overcome the first challenge by exploiting the simple structure of the binary decision problem, which allows us to classify errors into nine binary error flags under four broad categories. To obtain a distribution of GPT errors across categories, we then develop a GPT grader to efficiently process large-scale unstructured responses and determine the value of error flags within each category. Key inputs for the GPT grader include a reference answer using Bayes' rule, detailed grading rubrics, original experiment prompts and responses. We present our grading prompt in Appendix D.

We use a more advanced version, GPT o3-mini, to grade three less advanced models considered, ensuring the grader has superior performance in the binary classification tasks and general intelligence. We also manually reviewed 50 randomly sampled textual responses for each model to verify the GPT grader's performance. Overall, our independent cross-check confirmed the accuracy of GPT o3-mini's grading and classification of

errors in the responses from ChatGPT 3.5, 4 and 4o.[31]

6.1 Error Taxonomy

We focus on the textual responses from the 6-ball Wisconsin experiment. Table 7 presents the four broad categories and the error flags under each category.[32] The first type of error under Panel A examines whether GPTs understand the context and correctly interpret experimental parameters, including cage composition (i.e., the number of N balls in each cage), sample size (D), and the observed number of N balls (d).

The second type evaluates whether GPT subjects are "conceptually" Bayesian by checking if subjects take the prior information into consideration and if they use the sample information as inputs to their decision. Failure to use this information can explain choices that are consistent with conservatism and representativeness. We emphasize the *consideration* of both prior and likelihood in the second category, leaving the examination of *numerically correct* posterior calculation to the third category under Panel C.

In Panel C, we check whether the prior probability (π_A),[33] the likelihood probabilities ($f(d|p_A, D)$ and $f(d|p_B, D)$) and the posterior ($\frac{\pi_A f(d|p_A,D)}{\pi_A f(d|p_A,D)+(1-\pi_A)f(d|p_B,D)}$) or posterior odds ($\frac{\pi_A f(d|p_A,D)}{(1-\pi_A)f(d|p_B,D)}$) are correctly calculated.

There are a few points worth discussing. First, we explicitly instruct the GPT grader to allow for the omission of binomial coefficients when calculating the likelihood, as they will cancel out and do not affect the posterior calculation. Second, we find that some GPT subjects, instead of calculating the posterior probability $\Pi(A|d, \pi, p_A, p_B, D)$ as in equation 2, make decisions by calculating and comparing the product of prior and likelihood, $\pi \times f(d|p_A, D)$ and $(1 - \pi) \times f(d|p_B, D)$. This approach is consistent with Bayes' rule, and in such cases, the subject passes error flag 8. Third, similar to the identification challenge discussed in 3.1, it is sufficient for subjects to make correct decisions if the posterior is on the same side of 1/2 as the Bayesian posterior. However, since GPT subjects usually report the posterior probabilities, we apply a stricter grading

[31] In Appendix G, we report the error rates for the same set of 50 samples graded by GPT o3-mini and a human grader. Additionally, we include the grading results using the most advanced version of GPT o1, which, while offering slightly superior performance, is significantly more costly.

[32] To demonstrate the error flags, we provide excerpts from textual responses as example answers classified under each error flag in Appendix E.

[33] To calculate the prior probability, subjects must understand the process of rolling a 10-sided die and divide the specified range of results for Cage A by 10. While this is straightforward for humans, we occasionally find that GPT subjects use an incorrect numerator or denominator when calculating the prior probability for Cage A. See Appendix E for an example.

rubric by marking the subject as making an error in flag 8 if their calculated posterior (or posterior odds) was incorrect regardless of whether it leads to the same decision as the true posterior.

The final category under Panel D examines whether the final decision (Cage A or B) aligns with the previous reasoning, which substantiates the decision noise ε in the structural logit model. We instruct the GPT grader to read the overall reasoning of the textual responses and then predict the expected outcome based on the reasoning flow. Almost all GPT-4 and 4o subjects answer the problem by calculating the posterior or posterior odds. In such cases, we define error flag nine, final decision contradicting the previous reasoning, as choosing cage A when the posterior of Cage A is below $\frac{1}{2}$, or when $\pi \times f(d|p_A, D)$ is smaller than $(1-\pi) \times f(d|p_B, D)$. In other cases, we ask the GPT grader to explicitly predict the outcome based on the reasoning just before the final answer and compare whether this prediction is consistent with the subjects' report.

6.2 Grading Results

We apply the grading prompt to evaluate 500 randomly selected text responses for each GPT model, regardless of whether the final decision aligns with Bayes' rule. The output of the grading algorithm assigns a value to each error flag, with 1 indicating that the student's response contains a mistake. Table 7 presents the error rates for nine types of errors. We also report the fraction of responses inconsistent with the Bayes' rule at the bottom of the Table.[34]

[34]The sum of error rates across all categories doesn't necessarily match the fraction of incorrect responses for two reasons. First, errors are not mutually exclusive. If a subject ignores the prior (error flag 4 = YES), they will also have errors in calculating the prior (error flag 6 = YES) and subsequently the posterior (error flag 8 = YES). Second, since the final decision's consistency with Bayes' rule only requires comparing the posterior to $\frac{1}{2}$, an error in calculating the posterior does not necessarily lead to a mistake in the final decision.

Table 7: Error Distribution from GPT Responses

	Error Flag (Yes/No)	GPT 3.5 (%)	GPT 4 (%)	GPT 4o (%)
Panel A. Data read-in errors				
1	Error reading the compositions of the two cages	0.4	0.4	0.0
2	Error reading the number of balls drawn from the two cages	0.0	0.0	0.0
3	Error reading the outcome of the draws	0.0	0.0	0.0
Panel B. Errors in the application of Bayes Rule				
4	Ignoring the prior	82.4	1.4	0.6
5	Ignoring the likelihood	0.2	0.0	0.0
Panel C. Errors in computing the posterior probability				
6	Error calculating prior probability	83.4	2.2	0.6
7	Error calculating the likelihood	70.0	67.8	13.6
8	Error calculating the posterior (or posterior odds)	98.4	72.0	21.8
Panel D. Errors in the final decision				
9	Final decision contradicting the previous reasoning	4.4	3.0	4.6
	Fraction of responses inconsistent with Bayes' rule	35.2	14.2	9.2

Table 7 reinforces our conclusion about the rapid improvement from GPT-3.5 to GPT-4 and GPT-4o, as GPT-4 and GPT-4o make significantly fewer errors in almost all types. This not only reflects higher accuracy, but also indicates fewer mistakes in calculating posterior probabilities or other steps even when final decisions align with Bayes' rule.

Panel A shows that GPT subjects rarely make mistakes in reading experimental parameters, suggesting a good understanding of the experimental setup.

We observed a remarkable transformation from non-Bayesian to conceptual Bayesian reasoning from GPT-3.5 to GPT-4. More than 80% of GPT-3.5 decisions rely only on likelihood, lacking a Bayesian rationality even conceptually. This aligns with our findings in Section 5. Our manual review of the text responses shows that they often use representativeness heuristics, making decisions by matching observed patterns in the sample with the composition of the two cages, or simply comparing the likelihood of each cage being the sample's source. Interestingly, GPT-3.5 almost never ignores information from the sample, suggesting it is less prone to conservative bias. GPT-4 and GPT-4o almost always demonstrate at least "conceptual Bayesian" reasoning, either by explicitly writing down Bayes' formula or informally considering both prior and likelihood information.

Panel C highlights another major achievement in transitioning from "conceptual Bayesian" to "perfect Bayesian" when moving from GPT-4 to GPT-4o. In GPT-4o, 78% of responses

correctly calculate the posterior, compared to just 28% for GPT-4. GPT-3.5 struggles to calculate both the likelihood and prior probability, often due to overlooking prior information.

Panel D indicates that all models experience decision noise. Surprisingly, although GPT-4o generally outperforms GPT-4, its final decisions are more likely to be inconsistent with the calculated posterior, whether comparing it to $\frac{1}{2}$ or comparing the numerator and denominator of the posterior odds. To understand it, we delve deeper into the sample graded by the human grader, who also confirms this observation.[35] First, we find that GPT-4o is more likely to report the posterior as fractional numbers than GPT-4,[36] probably due to its ability to more accurately calculate the posterior, as shown in Panel C. However, the fractional numbers reported by GPT-4o, though precise, are more complex with more digits[37], making comparisons for the final decision more challenging.[38] Therefore, although GPT 4o can calculate the posterior more accurately, it may shoot itself in the foot when making final decisions by comparing these numbers.

6.3 Conclusion from Analysis of Textual Responses

We summarize our main finding from the analysis of textual responses here.

1. Analyzing the textual responses confirms the rapid improvement from GPT 3.5, to GPT-4 and 4o as more advanced GPTs make less mistakes in almost all error categories.
2. We observed a remarkable transformation from non-Bayesian to conceptual Bayesian reasoning from GPT-3.5 to GPT-4.
3. Transitioning from GPT-4 to GPT-4o represents another major leap from "conceptual Bayesian" to "perfect Bayesian," achieving the accurate calculation of a Bayesian posterior.

[35] The human grader reports an error rate of 14% for GPT-4o and 2% for GPT-4 regarding the decision inconsistency error flag. See Appendix G.

[36] Out of the 50 samples, 86% are reported as fractional numbers in GPT-4o, with the remainder as rounded decimals. This percentage decreases to 66% in GPT-4.

[37] See Appendix E for an example where the posterior for Cage A is calculated and reported as $\frac{2612736}{4200459}$, which equals 0.62 as a decimal. The GPT-4o model should have chosen Cage A, but instead, it chose Cage B.

[38] Out of 17 responses where GPT-4o reported posteriors as fractional numbers, 7 resulted in mistakes during the final decision. In contrast, none of the 7 cases with fractional posteriors from GPT-4 had such issues. We note that only 1 decision was inconsistent with the calculated posterior out of 43 cases for GPT-4 and 33 cases for GPT-4o when using rounded decimals.

7 Conclusion

This paper examines whether humans or ChatGPTs more closely resemble a Bayesian decision maker in a simple binary classification task. Although the task is straightforward, Bayesian rationality is fundamental in decision theory and broadly applicable to many important contexts.

We introduce a parsimonious structural logit model with unobserved heterogeneity to infer subjects' subjective beliefs. Our model demonstrates greater predictive power for human behaviors than those used in previous studies. Additionally, we develop a novel decision efficiency measure, allowing us to compare different groups with varying task difficulties, addressing limitations of the popular accuracy measure based on Bayes' rule consistency.

We estimate the structural logit model using data from human experiments by El-Gamal and Grether (1995), El-Gamal and Grether (1999), and Holt and Smith (2009), alongside our replication using various ChatGPTs. Our study differs from previous literature in two key ways. First, we account for significant heterogeneity across GPT versions, showing that conclusions vary due to rapid improvements in Bayesian rationality. We document the evolution from sub-human performance in early GPT-3.5 to near-human performance in GPT-4, and superior performance in GPT-4o. Second, we highlight considerable heterogeneity among human subjects, as revealed by our model. The most efficient humans can closely resemble Bayesian decision makers, comparable to the most advanced GPTs. We also find less unobserved heterogeneity in GPT subjects, particularly in more advanced versions.

As a first step in unraveling the decision mechanisms of GPTs, we leverage their ability to display reasoning in textual responses to analyze where they make mistakes. By exploring the simple structure of binary choice models, we categorize errors and provide clear guidance for GPT-4o graders to interpret these responses. Examining the extensive textual data reveals a remarkable shift from non-Bayesian to conceptual Bayesian reasoning from GPT-3.5 to GPT-4, with GPT-4o making another exceptional transition to accurately calculating posterior probabilities.

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Appendix A Proof of Lemma L2

This appendix provides the proof of identification of the structural logit model when $\sigma > 0$, Lemma L2 of section 3.1. We start by assuming that the agent's decision rule $P(A|\pi, n)$ is identified for all possible values of $\pi \in [0, 1]$ and all possible experiments involving draws from one of two bingo cages where the number of draws from these cages, and hence the possible values of n, can be arbitrarily large. In fact because we assume that the experimenter can run an arbitrary number of experiments and observe the subject outcomes, the experimenter has the freedom to design experiments with arbitrary values for the log-likelihood ratio LLR and the log-prior odds ratio LPR, so we assume these quantities can take any value between $-\infty$ and $+\infty$ via an appropriately designed set of experiments. Further, since the subject's decision rule depends on (π, n) via the quantities $(\text{LPR}(\pi), \text{LLR}(n))$ and the experimenter has the freedom to design experiments that allow LPR and LLR to take on any values in R^2, we can treat the subject's decision rule as a known function of two continuous arguments, (LPR, LLR), which constitute "sufficient statistics" for the subject's posterior beliefs, and hence for the subject's decision rule. In short, we can assume that the subject's decision rule is a known conditional probability of the form $P(A|\text{LPR}, \text{LLR})$ mapping R^2 into $[0, 1]$.

Now, let the subject's true decision rule be given by the function

$$P(A|\text{LPR}, \text{LLR}, \sigma^*, \beta^*) = \frac{1}{1 + \exp\{[2\Pi_s(A|\text{LPR}, \text{LLR}, \beta^*) - 1]/\sigma^*\}}, \tag{23}$$

and by assumption, this function is identified, i.e. its value is known for any pair $(\text{LPR}, \text{LLR}) \in R^2$. Now suppose there is some other structural logit that is observationally equivalent to the true one. That is, suppose there is a function $P(A|\text{LPR}, \text{LLR}, \sigma, \beta)$ such that

$$P(A|\text{LPR}, \text{LLR}, \sigma^*, \beta^*) = P(A|\text{LPR}, \text{LLR}, \sigma, \beta), \quad \forall(\text{LPR}, \text{LLR}) \in R^2. \tag{24}$$

We will now that that if this is true, then it must be the case that $\sigma = \sigma^*$, and $\beta = \beta^*$. That is, the parameters of the structural logit model are identified.

We can show this under a *full support condition* which is a version of an "identification at infinity" type of argument. The full support condition implies that the experimenter can conduct sufficient experimentation on subjects that the log-prior ratio LPR takes on any value on the real line. It follows that by taking the limit as $\text{LPR} \to \infty$ we have $\Pi_s(A|\text{LPR}, \text{LLR}) \to 0$. Since equation (24) holds for all $(\text{LPR}, \text{LLR}) \in R^2$ and is a continuous function of these variables, it follows that the equality must hold in the limit so we have

$$\frac{1}{1 + \exp\{1/\sigma^*\}} = \frac{1}{1 + \exp\{1/\sigma\}}, \tag{25}$$

which implies that $\sigma^* = \sigma$, so this parameter is identified. Using this result we can immediately conclude from equations (23) and (24) that

$$\Pi_s(A|\text{LPR}, \text{LLR}, \beta^*) = \Pi_s(A|\text{LPR}, \text{LLR}, \beta). \tag{26}$$

Since $\Pi_s(A|\text{LPR}, \text{LLR}, \beta) = 1/(1 + \exp\{\beta_0 + \beta_1\text{LPR} + \beta_2\text{LLR}\}$, it follows that we have

$$\beta_0^* + \beta_1^*\text{LPR} + \beta_2^*\text{LLR} = \beta_0 + \beta_1\text{LPR} + \beta_2\text{LLR}, \quad \forall(\text{LPR}, \text{LLR}) \in R^2. \tag{27}$$

Let LPR = LLR = 0 (which is possible due to the full support assumption). If follows from equation (27) that $\beta_0^* = \beta_0$. Next, set LLR = 0 and LPR = 1, and it follows that $\beta_1^* = \beta_1$. Finally, set LLR = 1 and LPR = 0, and it follows that $\beta_2^* = \beta_2$. We conclude that the parameters $(\sigma^*, \beta_0^*, \beta_1^*, \beta_2^*)$ of the structural logit model are identified.

Appendix B Algorithm for Data Collection from LLMs

Algorithm 1 Data Collection and Processing

Initialize data collection specifications based on the experiment specified by `run_name`, generate textual prompts and model settings, and save to disk
`'send'` prompts and model settings to OpenAI via API
while any request is **not** completed **do**
    for every request **do**
        `'retrieve'` the request's status
        if the request has been completed **and not** retrieved **then**
            Retrieve the responses and save to disk
        else if the request has failed **then**
            `'resend_failed'` request(s)
        else if
            thencontinue
        end if
    end for
    `'finalize'` the responses, which includes:
        Align responses with the original prompts
        Parse the responses into the final answers like Cage A or B or a numerical value
        Append metadata and informational columns
        Save different formats of the collected data to disk
        Check for invalid responses that cannot be parsed into a final answer
    if any response is invalid **then**
        `'resend_invalid'` prompts and model settings to OpenAI via API
    end if
end while

Appendix C Prompts used to collect data from LLMs

In this section, we provide our prompt to replicate using LLMs the two experiments at the University of Wisconsin-Madison and the two experiments reported in Holt and Smith (2009).

There are no `developer` / `system` messages, only one `user` message for each chat completion request for each trial. The exact numerical values will reflect the specifications of the actual trial, the followings are examples.

C.1 Wisconsin

For the experiment that allows for `reasoning`, the example `user` message is:

```
You are participating in a decision-making experiment, where you
   can earn money based on the number of correct decisions you
   make.

There are two identical bingo cages, Cage A and Cage B, each
   containing 6 balls. Cage A contains 4 balls labeled "N" and 2
   balls labeled "G", while Cage B contains 3 balls labeled "N"
   and 3 balls labeled "G".

A 10-sided die is used to determine which of the two cages will
   be used to generate draws. If a random roll of the die shows 1
    through 3, I will use Cage A; if it shows 4 through 10, I
   will use Cage B. You will not know the outcome of the roll of
   the die or which cage I use.

Once a cage is chosen at random based on the roll of the die, it
   is used to generate draws with replacement.

I have drawn a total of 6 balls with replacement. The result is 3
    "N" balls and 3 "G" balls.
After observing this outcome, which cage do you think generated
   the observations? Your decision is correct if the balls were
   drawn from that cage.

YOU ARE WELCOME TO ALSO DESCRIBE YOUR REASONING, BROKEN INTO
   SEPARATE STEPS, TO EXPLAIN HOW YOU ARRIVED AT YOUR FINAL
   ANSWER.
Please state your answer in the following format at the end.
"Final answer: Cage A." or "Final answer: Cage B.".
```

For the experiment that prohibits `reasoning`, the last section of the `user` message is substituted with:

```
PLEASE JUST REPORT YOU FINAL ANSWER AND DO NOT PROVIDE ANY
   REASONING AS TO HOW YOU ARRIVED AT YOUR FINAL ANSWER.
Please state your answer in the following format.
"Final answer: Cage A." or "Final answer: Cage B.".
```

C.2 Holt and Smith

For the experiment that allows for `reasoning`, the example `user` message is:

```
This is an experiment in the economics of decision making.
   Various agencies have provided funds for the experiment. Your
   earnings will depend partly on your decisions and partly on
   chance. If you are careful and make good decisions, you may
   earn a considerable amount of money, which will be paid to you
   , privately, in cash, at the end of the experiment. In
   addition to the money that you earn during the experiment, you
    will also receive $6. This payment is to compensate you for
   showing up today.

This experiment involves two stages. In stage 1 we will show you
   some information including the result of a drawing of 1 ball
   from one of two possible cages, each containing different
   numbers of light and dark balls. Then at the start of stage 2
   you will report a number P between 0 and 1. After your report,
    we will draw a random number U that is equally likely to be
   any number between 0 and 1. Your payoff from this experiment
   will either be $1000 or $0 depending on your report P and the
   random number U.

Let's describe the two stages in more detail now. In stage 1 we
   will show you 1 ball that are drawn at random from one of two
   possible urns labelled A and B.

Urn A contains 2 light balls and 1 dark ball.
Urn B contains 1 light ball and 2 dark balls.

We select the urn, A or B, from which we draw the sample of 1
   ball by the outcome of throwing a 6 sided die.
We do not show you the outcome of this throw of the die but we do
    tell you the rule we use to select urn A or B.

If the outcome of the die throw is 1 to 3 we select urn A.
If the outcome of the die throw is 4 to 6, we use urn B to draw
   the random sample of 1 ball.

Once you see the outcome of the sample of 1 ball, stage 1 is over
    and stage 2 begins.

At the start of stage 2 we ask you to report a number P between 0
    and 1. Your payoff from this experiment depend on another
   random number, which we call U, which we draw after you report
    the number P. We draw the random number U in a way that every
    possible number between 0 and 1 has an equal chance of being
   selected.
```

```
Here is how you will be paid from participating in this
   experiment. There are two possible cases:

Case 1. If the number U is less than or equal to P then you will
   receive $1000 if the sample of 1 ball we showed you in stage 1
    was from urn A and $0 otherwise.
Case 2. If the number U is between the number P you report and 1,
    you will receive $1000 with probability equal to the realized
    value of U, but with probability 1-U you will get $0.

OK, this is the setup. Let's now start begin this experiment,
   starting with stage 1.

We have tossed the die (the outcome we don't show to you) and
   selected one of these urns according to the rule given above (
   i.e. urn A if the die throw was 1 to 3, and urn B otherwise).
   We have drawn 1 ball from the selected urn and the outcome is
   D, i.e., Dark.

Now, we are at stage 2 where we are asking you, given the
   information from stage 1 to report a number P between 0 and 1
   that in conjunction with the random number U will determine if
    you get either $1000 or $0 according to the rule given in
   cases 1 and 2 above.

Please report a number P between 0 and 1 that maximizes your
   probability of winning $1000 in this experiment.

YOU ARE WELCOME TO ALSO DESCRIBE YOUR REASONING, BROKEN INTO
   SEPARATE STEPS, TO EXPLAIN HOW YOU ARRIVED AT YOUR FINAL
   ANSWER P.
Please state your answer in the following format at the end.
Final answer: [your P value here].
```

For the experiment that prohibits `reasoning`, the last section of the `user` message is substituted with:

```
PLEASE JUST REPORT P AND DO NOT PROVIDE ANY REASONING AS TO HOW
   YOU ARRIVED AT THE VALUE P.
Please state your answer in the following format.
Final answer: [your P value here].
```

Appendix D Grading Prompts

We provide a grading prompt that evaluates GPT-generated textual responses and identifies errors. First, we introduce the grading task and the role of the grader to GPT-01. Next, we present the experiment design prompt along with the original responses from GPT students. The key input for our grading prompt is a meticulously crafted step-by-step solution to the binary choice problem, which includes calculating the prior probability, determining the likelihood, applying Bayes' rule to find the posterior probability, and making a final decision based on the posterior. This solution is crucial as a benchmark for the GPT grader to understand the binary decision task and identify errors in the student responses. We then provide detailed grading rubrics that allow the GPT grader to classify different error types. Finally, the prompt specifies the format for outputting error flags. We develop Python code to loop over different student responses, parameters of trials, and the corresponding benchmark solutions.

General Instruction about the Role of the Grader

```
You are an expert and detailed grader for this task:
```

Introduce the Experiment Prompt

```
You are participating in a decision-making experiment, where you
   can earn money based on the number of correct decisions you
   make.

There are two identical bingo cages, Cage A and Cage B, each
   containing 10 balls. Cage A contains 4 balls labeled "N" and 6
    balls labeled "G", while Cage B contains 6 balls labeled "N"
   and 4 balls labeled "G".

A 10-sided die is used to determine which of the two cages will
   be used to generate draws. If a random roll of the die shows 1
    through 6, I will use Cage A; if it shows 7 through 10, I
   will use Cage B. You will not know the outcome of the roll of
   the die or which cage I use.

Once a cage is chosen at random based on the roll of the die, it
   is used to generate draws with replacement.

I have drawn a total of 7 balls with replacement. The result is 3
    "N" balls and 4 "G" balls.
After observing this outcome, which cage do you think generated
   the observations? Your decision is correct if the balls were
   drawn from that cage.

YOU ARE WELCOME TO ALSO DESCRIBE YOUR REASONING, BROKEN INTO
   SEPARATE STEPS, TO EXPLAIN HOW YOU ARRIVED AT YOUR FINAL
   ANSWER.
Please state your answer in the following format at the end.
"Final answer: Cage A." or "Final answer: Cage B.".
```

Student textual responses

```
Based on the outcome of 3 "N" balls and 4 "G" balls, here is my
   reasoning broken down into steps:

Step 1: Calculate the probability of getting 3 "N" balls and 4 "G
   " balls from each cage.

- Probability of getting 3 "N" balls and 4 "G" balls from Cage A:
    (4/10)^3 * (6/10)^4 = 0.00324
- Probability of getting 3 "N" balls and 4 "G" balls from Cage B:
    (6/10)^3 * (4/10)^4 = 0.00307

Step 2: Compare the probabilities of the two cages.

- The probability of getting the observed outcome is slightly
   higher for Cage A than Cage B.

Step 3: Consider the initial probabilities of each cage being
   chosen.

- Cage A has a 6/10 chance of being chosen, while Cage B has a
   4/10 chance.

Step 4: Combine the probabilities of the observed outcome and the
    initial probabilities of the cages being chosen.

- Probability of getting the observed outcome from Cage A:
   0.00324 * 0.6 = 0.00194
- Probability of getting the observed outcome from Cage B:
   0.00307 * 0.4 = 0.00123

Step 5: Compare the combined probabilities.

- The combined probability of getting the observed outcome is
   higher for Cage A than Cage B.

Final answer: Cage A.
```

Step-by-step Solution Benchmark

```
Step 1: Prior from die roll.
  p(A) = 3/5 = 0.6000;  p(B) = 2/5 = 0.4000
  Prior ratio p(A)/p(B) = 3/2 = 1.500000
Step 2: Cage compositions.
  Cage A: p(NA) = 2/5 = 0.4000, p(GA) = 3/5 = 0.6000
  Cage B: p(NB) = 3/5 = 0.6000, p(GB) = 2/5 = 0.4000
Step 3: Observed outcome & binomial likelihood.
  Observed: 3 'N', 4 'G' (total 7).
  L(A) = 4536/15625 = 0.290304, L(B) = 3024/15625 approx 0.193536
  Likelihood ratio L(A)/L(B) = 3/2 = 1.500000
```

```
Step 4: Posterior components & probabilities (fraction & decimal)
   .
  p(A)*L(A) = 13608/78125 approx 0.174182
  p(B)*L(B) = 6048/78125 approx 0.077414
  Post(A)   = 9/13 = 0.692308
  Post(B)   = 4/13 = 0.307692
  Posterior ratio Post(A)/Post(B) = 9/4 = 2.250000
Step 5: Decision.
  Final answer: Cage A.
```

Grading Rubrics

```
Evaluate the student's answer based on the following criteria:

Part I. Did They Make a Mistake When Reading the Data?

Instructions: For (1), (2), and (3), we want to see if the
   student understands the basic experimental setup and correctly
    incorporates the trial parameters into their reasoning. Focus
    on whether they read the relevant information accurately, not
    on how they use it later. For example, if a student correctly
    identifies the number of N balls in cages A and B but makes a
    comparison error later, you should still answer YES if the
   criterion is reading the number of N balls correctly.

(1) Cage Composition: Do they explicitly mention or implicitly
   acknowledge the number of N and G balls in each cage? Answer
   Yes or No.

(2) Draw Count: Do they explicitly mention or implicitly
   acknowledge the total number of balls in the sample? Answer
   Yes or No.

(3) Observed Data: Do they explicitly mention or implicitly
   acknowledge the number of N draws from the sample? Answer Yes
   or No.

Part II. Are they conceptually Bayesian?

Instructions: A Bayesian decision maker should consider both
   prior information (the announced probability of using a cage)
   and posterior information (the likelihood that the sample was
   drawn from a cage). Criteria (4) and (5) assess whether the
   student incorporates both prior and posterior information in
   their reasoning. We are not looking for explicit numerical
   calculations, but both types of information should be part of
   their reasoning process.

(4) Ignoring Prior: Do they make a decision using only the
   likelihood or observed data, ignoring the prior conceptually?
```

```
   Answer Yes or No.

(5) Ignoring Likelihood: Do they make a decision using only the
   announced probability of using cage A, ignoring the sample
   information conceptually? Answer Yes or No.

Part III. Can they correctly calculate the Bayesian posterior
   probability?

Instructions: To answer correctly, the student should apply Bayes
   ' rule and calculate the posterior probability accurately.
   This involves three steps:

Prior Probability: Calculate the prior probability that the
   sample is drawn from cage A and B, based on the announced
   probability in the experiment.
Likelihood: Calculate the likelihood that a sample is drawn from
   each cage, using the number of N draws in the sample and the
   cage composition.
Posterior Probability: Either calculate the posterior probability
    or compute the product of likelihood and prior for each cage.
Note: Values are equal if they round to the same number at two
   decimal places. For example, 0.333 and 0.33 should be treated
   as the same. If the student doesn't attempt the calculation or
    leaves it incomplete, answer No. If interrupted, also answer
   No.

(6) Prior Computed: Do they calculate the prior probability for
   each cage correctly? Answer Yes or No.

(7) Likelihood Computed: Do they calculate the likelihood
   correctly? Answer Yes or No. Note, that omitting the binomial
   coefficient is acceptable, as it is a constant for both cages.

(8) Posterior Computed: Do they apply Bayes' rule and calculate
   the posterior probability correctly? Alternatively, answer Yes
    if they correctly compute and compare the product of
   likelihood and prior probability. Answer Yes or No.

Part IV. Do they make a final decision that is consistent with
   their reasoning process? Instructions: The student should
   reach a conclusion based on their reasoning process, and the
   final answer should align with that conclusion. You should
   examine the student's reasoning and predict what they should
   report (e.g., cage A or cage B), then compare it to their
   actual report. Provide YES if they are consistent.

Here are two examples of inconsistency.
```

```
A student calculates the posterior probability of cage A as 2/3.
   Since 2/3 is greater than 1/2, they should report cage A but
   instead report cage B.
A student finds the product of likelihood and prior to be 2/3 for
    cage A and 1/3 for cage B. Since 2/3 is greater than 1/3,
   they should choose cage A but report cage B.
(9) Inconsistency: Based on the student's reasoning process and
   decision rule, determine what the final decision should be. Is
    the decision they actually make consistent with their
   reasoning? Answer Yes or No.
Note, that we are looking for cases where the student has reduced
    the answer to decimals or simple fractions, and decides
   against what they should have decided.
We do want to penalize fraction-to-decimal conversion errors if
   they clearly lead the student to the wrong final choice.
If the student never attempts a numeric calculation or fails to
   provide a final decision at all, also answer No.
```

Output Instructions

```
Provide detailed reasoning first and then at the end output a
   JSON. Your answers must use: Yes = 1, No = 0.

    {{
      "1_correct_cage_composition": 1/0,
      "2_correct_draw_count_and_with_replacement": 1/0,
      "3_correct_use_of_observed_data": 1/0,
      "4_conceptual_ignore_prior_likelihood_only": 1/0,
      "5_conceptual_ignore_likelihood_prior_only": 1/0,
      "6_compute_prior_exact": 1/0,
      "7_compute_likelihood_exact": 1/0,
      "8_compute_posterior_exact": 1/0,
      "9_decision_consistency": 1/0
    }}
```

Appendix E Examples of Textual Responses under Error Flags

In this section, we excerpt from representative examples of *students*' original textual responses, classified by the GPT grader under each category, along with the *grader*'s responses explaining the reasoning behind its classifications. We don't present examples for Error Flags 2, 3 and 5 because they are rarely, if any, detected.

1. Cage Composition

Student

Based on the given information, here is my reasoning broken into separate steps:
Step 1. Since there are more N balls in Cage A than in Cage B, and the outcome after 7 draws is 4 N balls and 3 G balls, it's more likely that the draws came from Cage A because there are more N balls available in that cage.
Step 2. The probability of getting a N ball from Cage A is $4/10 = 0.4$, and the probability of getting a G ball from Cage A is $6/10 = 0.6$.
Step 3. The probability of getting a N ball from Cage B is $6/10 = 0.6$, and the probability of getting a G ball from Cage B is $4/10 = 0.4$.
Step 4. Considering the outcomes of 4 N balls and 3 G balls after 7 draws, it is more probable that the draws came from Cage A because the probability of drawing a N ball from Cage A (0.4) is higher than drawing a N ball from Cage B (0.6).
Step 5. Therefore, my final answer is: Final answer: Cage A.

Grader

The student states there are more N balls in Cage A than in Cage B. This is incorrect because Cage A actually has 4 N balls and Cage B has 6 N balls. Thus, they misread the cage composition.

4. Ignoring Prior

Student

Step 1: Calculate the probability of getting 3 "N" balls and 3 "G" balls from each cage.
- For Cage A: The probability of drawing an "N" ball from Cage A is $4/6 = 2/3$. The probability of drawing a "G" ball from Cage A is $2/6 = 1/3$. The probability of getting 3 "N" balls and 3 "G" balls from Cage A is $(2/3)^3 * (1/3)^3 = 8/729$.
- For Cage B: The probability of drawing an "N" ball from Cage B is $3/6 = 1/2$. The probability of drawing a "G" ball from Cage B is also $1/2$. The probability of getting 3 "N" balls and 3 "G" balls from Cage B is $(1/2)^3 * (1/2)^3 = 1/64$.
Step 2: Compare the probabilities of the outcomes from each cage.
- The probability of getting 3 "N" balls and 3 "G" balls is higher for Cage A ($8/729$) compared to Cage B ($1/64$).
Step 3: Make a decision based on the higher probability outcome.
Final answer: Cage A.

Grader

The student never mentioned or used the die roll prior probabilities (3/10 for Cage A and 7/10 for Cage B) and made the decision solely based on the likelihood calculation. Thus they did ignore prior information.

6. Prior Computed

Student

Step 1: Calculate the probability of drawing 6 "N" balls from each cage: For Cage A: $(4/6)^6 = 0.012$, For Cage B: $(3/6)^6 = 0.015$
Step 2: Compare the probabilities: - The probability of drawing 6 "N" balls from Cage B is slightly higher than from Cage A.
Step 3: Consider the initial composition of the cages: - Cage A has more "N" balls compared to Cage B, which could make it more likely to draw 6 "N" balls from that cage.
Final answer: Cage A.

Grader

The student does not calculate or even consider the prior probabilities (the announced cage probabilities).

7. Likelihood Computed

Student

Step 1: Calculate the probability of getting 2 "N" balls and 5 "G" balls from each cage.
For Cage A: P(2 "N" balls and 5 "G" balls from Cage A) = P(2 "N" balls from Cage A) * P(5 "G" balls from Cage A) = $(4/10)^2 * (6/10)^5 \approx 0.0288$
For Cage B: P(2 "N" balls and 5 "G" balls from Cage B) = P(2 "N" balls from Cage B) * P(5 "G" balls from Cage B) = $(6/10)^2 * (4/10)^5 \approx 0.013824$
Step 2: Compare the probabilities calculated above.
Since the probability of getting 2 "N" balls and 5 "G" balls from Cage A is higher than from Cage B, it is more likely that the observations were generated from Cage A.
Final answer: Cage A.

Grader

Step 3: Observed outcome and binomial likelihood. Observed: 2 'N', 5 'G' (total 7). L(A) = 20412/78125 $\approx$ 0.261274 L(B) = 6048/78125 $\approx$ 0.077414 Likelihood ratio L(A)/L(B) = 27/8 = 3.375000

The student attempts to compute the likelihoods, but they did not include the proper terms (or at least did not compute using the correct values as the proper binomial coefficient is omitted, and the numerical values do not match the correct

likelihood calculation). Moreover, their computed numbers do not lead to the correct likelihood ratio.

8. Posterior Computed

Grader

To determine which cage is more likely to have generated the observed outcome, we can use Bayes' theorem. The steps for reasoning through this problem are as follows:
Step 1: Determine Prior Probabilities
- **Probability of using Cage A**: Since the die shows 1 through 4 for Cage A, $P(A) = \frac{4}{10} = 0.4$.
- **Probability of using Cage B**: Since the die shows 5 through 10 for Cage B, $P(B) = \frac{6}{10} = 0.6$.

Step 2: Determine Likelihoods - **Likelihood of drawing 5 "N" and 1 "G" from Cage A**: - Probability of drawing "N" from Cage A: $\frac{4}{6} = \frac{2}{3}$.
- Probability of drawing "G" from Cage A: $\frac{2}{6} = \frac{1}{3}$.
- Probability of the observed outcome from Cage A:

$$\left(\frac{2}{3}\right)^5 \times \left(\frac{1}{3}\right)^1 = \frac{32}{243}.$$

- **Likelihood of drawing 5 "N" and 1 "G" from Cage B**: - Probability of drawing "N" from Cage B: $\frac{3}{6} = \frac{1}{2}$.
- Probability of drawing "G" from Cage B: $\frac{3}{6} = \frac{1}{2}$.
- Probability of the observed outcome from Cage B:

$$\left(\frac{1}{2}\right)^5 \times \left(\frac{1}{2}\right)^1 = \frac{1}{64}.$$

Step 3: Apply Bayes' Theorem - We want to compare the posterior probabilities $P(A \mid \text{observed outcome})$ and $P(B \mid \text{observed outcome})$.

Using Bayes' theorem:

$$P(A \mid \text{observed outcome}) \propto P(\text{observed outcome} \mid A) \times P(A) = \frac{32}{243} \times 0.4$$

$$P(B \mid \text{observed outcome}) \propto P(\text{observed outcome} \mid B) \times P(B) = \frac{1}{64} \times 0.6$$

Calculate these products: - For Cage A: $\frac{32}{243} \times 0.4 = \frac{12.8}{243}$. - For Cage B: $\frac{1}{64} \times 0.6 = \frac{0.6}{64}$.

Step 4: Compare Posterior Probabilities To compare the likelihoods, it is easier to

compare the ratios: - The ratio of the likelihood for Cage A to Cage B is:

$$\frac{\frac{12.8}{243}}{\frac{0.6}{64}} = \frac{12.8 \times 64}{243 \times 0.6} = \frac{819.2}{145.8} \approx 5.62$$

Since the ratio is greater than 1, Cage A is more likely to have produced the observed outcome.

Conclusion Given that the posterior probability is higher for Cage A, it is more likely that Cage A generated the observations.
Final answer: Cage A.

Grader

The student applies Bayes rule by multiplying the (miscomputed) likelihood by the correct prior. Although they combine the two pieces of information, the incorrect likelihood for Cage A (and thus the resulting ratio of products) leads to a numerical error in the posterior computation. Therefore, the computation is not exact.

9. Decision Inconsistency

Student

To determine which cage is more likely to have generated the result of 3 "N" balls and 3 "G" balls, let's analyze the problem step by step.
Step 1: Determine the Probability of Choosing Each Cage - The probability of choosing Cage A is determined by the roll of the die showing 1 through 7. So, $P(A) = \frac{7}{10} = 0.7$. - The probability of choosing Cage B is determined by the roll of the die showing 8 through 10. So, $P(B) = \frac{3}{10} = 0.3$.
Step 2: Calculate the Probability of the Observed Result for Each Cage
Cage A: - Probability of drawing "N" from Cage A $= \frac{4}{6} = \frac{2}{3}$. - Probability of drawing "G" from Cage A $= \frac{2}{6} = \frac{1}{3}$. - To get exactly 3 "N" and 3 "G" in 6 draws, we use the binomial probability formula:

$$P(3N, 3G|A) = \binom{6}{3} \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^3$$

$$= 20 \times \left(\frac{8}{27}\right) \times \left(\frac{1}{27}\right)$$

$$= 20 \times \frac{8}{729}$$

$$= \frac{160}{729}$$

Cage B: - Probability of drawing "N" from Cage B $= \frac{3}{6} = \frac{1}{2}$. - Probability of

drawing "G" from Cage B = $\frac{3}{6} = \frac{1}{2}$. - To get exactly 3 "N" and 3 "G" in 6 draws:

$$P(3N, 3G|B) = \binom{6}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^3$$

$$= 20 \times \left(\frac{1}{8}\right) \times \left(\frac{1}{8}\right)$$

$$= 20 \times \frac{1}{64}$$

$$= \frac{20}{64} = \frac{5}{16}$$

Step 3: Use Bayes' Theorem to Update Beliefs
We need to calculate $P(A|3\text{N}, 3\text{G})$ and $P(B|3\text{N}, 3\text{G})$. According to Bayes' Theorem:

$$P(A|3\text{N}, 3\text{G}) = \frac{P(3\text{N}, 3\text{G}|A) \times P(A)}{P(3\text{N}, 3\text{G})}$$

$$P(B|3\text{N}, 3\text{G}) = \frac{P(3\text{N}, 3\text{G}|B) \times P(B)}{P(3\text{N}, 3\text{G})}$$

The denominator $P(3\text{N}, 3\text{G})$ is the same for both, so we can compare the numerators directly:

$$P(A|3\text{N}, 3\text{G}) \propto \frac{160}{729} \times 0.7$$

$$P(B|3\text{N}, 3\text{G}) \propto \frac{5}{16} \times 0.3$$

Calculating these: - $P(A|3\text{N}, 3\text{G}) \propto \frac{160}{729} \times 0.7 = \frac{112}{729}$ - $P(B|3\text{N}, 3\text{G}) \propto \frac{5}{16} \times 0.3 = \frac{1.5}{16} = \frac{1.5\times729}{16\times729} = \frac{1.5\times729}{11664}$
Comparing these values, $P(B|3\text{N}, 3\text{G})$ is greater.
Conclusion
The observed result is more likely with Cage B than with Cage A, given the probabilities calculated. Therefore, the draws are more likely to have come from Cage B.
Final answer: Cage B.

Grader

Step 1: Prior from die roll.
p(A) = 7/10 = 0.7000; p(B) = 3/10 = 0.3000
Prior ratio p(A)/p(B) = 7/3 ≈ 2.333333
Step 2: Cage compositions.
Cage A: p(N—A) = 2/3≈ 0.6667, p(G—A) = 1/3 ≈ 0.3333
Cage B: p(N—B) = 1/2 = 0.5000, p(G—B) = 1/2 = 0.5000
Step 3: Observed outcome and binomial likelihood. Observed: 3 'N', 3 'G' (total 6).
L(A) = 160/729 ≈ 0.219479, L(B) = 5/16 = 0.312500

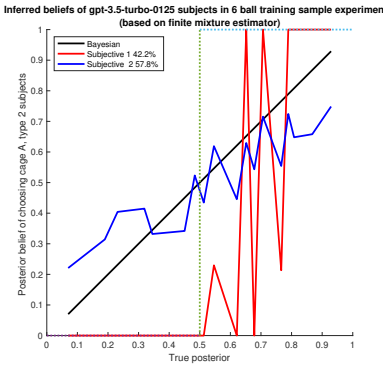
Likelihood ratio L(A)/L(B) = 512/729 ≈ 0.702332
Step 4: Posterior components and probabilities (fraction and decimal).
p(A)*L(A) = 112/729 ≈ 0.153635, p(B)*L(B) = 3/32 ≈ 0.093750
Post(A) = 3584/5771 ≈ 0.621036, Post(B) = 2187/5771 ≈ 0.378964
Posterior ratio Post(A)/Post(B) = 3584/2187 ≈ 1.638775
Step 5: Decision.
Final answer: Cage A.

Because the correctly computed product for Cage A (≈ 0.1537) is higher than that for Cage B (≈ 0.09375), the decision should have been Cage A. The student's final answer Cage B is inconsistent with the calculations.

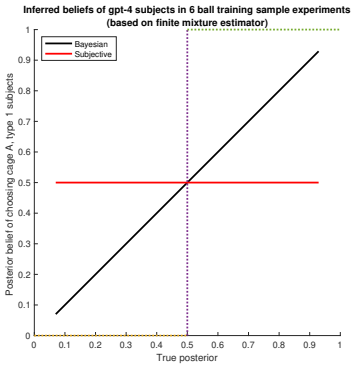
Appendix F Additional Figures

Figure F1: Estimated subjective posterior beliefs of GPT Subjects: 6-ball Experiments

(a) GPT 3.5



(b) GPT 4



(c) GPT 4o

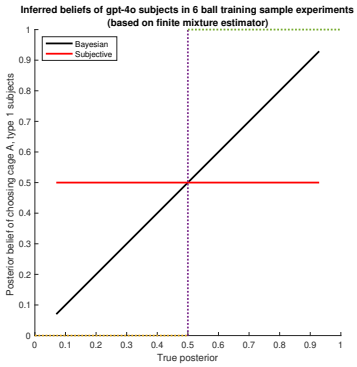
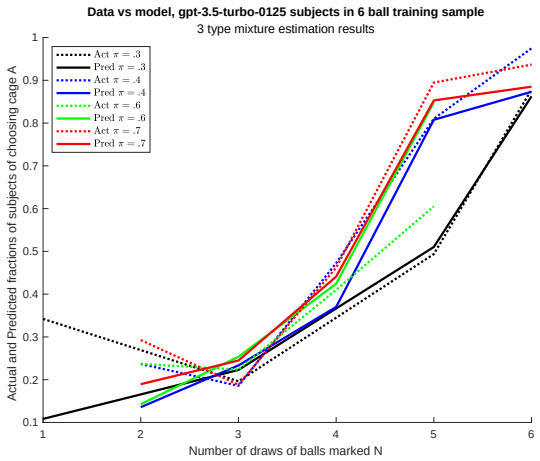
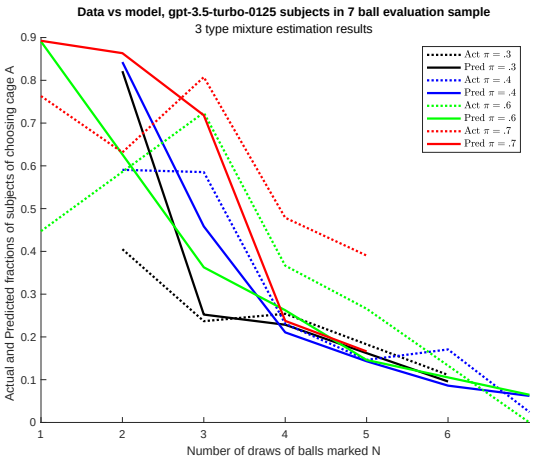


Figure F2: Predicted vs Actual CCPs in Training and Evaluation Samples





Appendix G Validation of the Automated GPT Grader

The analysis presented in Section 6 utilizes an automated approach, employing a "teacher" GPT grader to evaluate the textual reasoning provided by the "student" GPT models (GPT-3.5, GPT-4, and GPT-4o). To ascertain the reliability of this method, a validation exercise was conducted. The validation methodology involved a randomly selected subset of 150 textual responses (50 for each student model: GPT-3.5, GPT-4, GPT-4o) from the 6-ball Wisconsin experiment dataset. These responses were independently evaluated by three graders: a human co-author expert in the task, the GPT-o1 model, and the primary GPT-o3-mini grader. All graders utilized the finalized grading prompt detailed in Appendix D. Each response was assessed against the 9 binary grading flags described in Section 6.1, which were subsequently grouped into four conceptual panels: Panel A (Data Read-in), Panel B (Bayes Rule Application), Panel C (Posterior Calculation), and Panel D (Final Decision Consistency). Aggregate panel error rates were calculated, where a panel error was recorded if any underlying flag indicating an error within that panel was triggered for a given response. The error rates reported represent the percentage of the 50 responses for each student model flagged with an error in that panel.

Table 8 presents the comparative panel error rates generated by the three graders on the 50-sample validation subset. The results demonstrate a high degree of concordance between the primary automated grader (GPT-o3-mini), the advanced GPT-o1 grader, and the human expert, particularly for Panels A and B, where error rates are nearly identical. This confirms the automated graders' proficiency in identifying fundamental comprehension and conceptual errors related to Bayes' rule application.

Table 8: Comparison of Panel Error Rates (%) Across Graders (N=50 per student model)

Student Model	Panel	Error Category	GPT-o3-mini Grader	GPT-o1 Grader	Human Grader
GPT-3.5	Panel A	Data read-in errors	0	2	[2]
	Panel B	Bayes Rule application errors	76	76	[76]
	Panel C	Posterior calculation errors	96	100	[98]
	Panel D	Final decision inconsistency errors	6	10	[10]
GPT-4	Panel A	Data read-in errors	0	0	[0]
	Panel B	Bayes Rule application errors	4	4	[4]
	Panel C	Posterior calculation errors	64	74	[82]
	Panel D	Final decision inconsistency errors	0	0	[2]
GPT-4o	Panel A	Data read-in errors	0	0	[0]
	Panel B	Bayes Rule application errors	0	0	[0]
	Panel C	Posterior calculation errors	20	18	[22]
	Panel D	Final decision inconsistency errors	8	12	[14]

Note: Values represent the percentage of the 50 responses flagged with at least one error within the specified panel by the respective grader.

Minor discrepancies arise in Panels C and D, which involve assessing complex numerical calculations and logical consistency. For Panel C (Posterior Calculation), particularly with GPT-4 responses, GPT-o1 aligns more closely with the human grader (74% error

rate) than GPT-o3-mini does (64% error rate, vs. 82% for human). This suggests GPT-o1 has a superior, albeit still imperfect, ability to evaluate intricate numerical steps. Similarly, for Panel D (Final Decision Inconsistency), GPT-o1 again tracks human judgment more closely, especially for GPT-4o (12% vs. 14% for human, compared to 8% for o3-mini). These differences likely stem from the challenges LLMs face in precisely evaluating complex fraction comparisons and conversions, a task where GPT-o1 demonstrates marginally better performance.

Despite these minor variations in evaluating complex numerical reasoning, the overall agreement across graders is substantial. Importantly, the core qualitative findings reported in Section 6 are robustly identified by all graders. This includes the transition from conceptual errors (Panel B) dominating in GPT-3.5 to calculation errors (Panel C) being more prevalent in GPT-4, followed by significant improvement in calculation accuracy (Panel C) for GPT-4o. This validation exercise confirms that GPT-o3-mini serves as a reliable primary grader for the large-scale textual analysis.